Reflection Groups

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Course notes from 2017

Contents

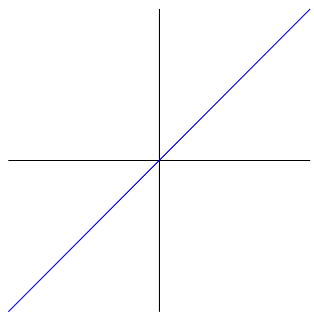
1 Sources

The main source for these notes is the book *Finite Reflection Groups* by Benson and Grove, Springer Verlag, Graduate Texts in Mathematics 99.

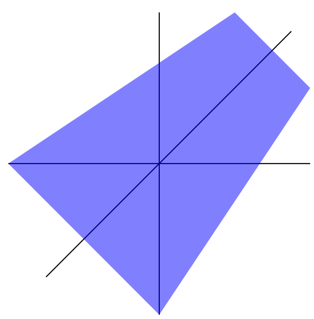
2 Reflections

This course is about finite reflection groups. By definition, these are the finite groups generated by reflections, and to begin the course I will define reflections. To define reflections, we need to first recall some material about vector spaces. Throughout the course, we will often let $V = \mathbb{R}^n$, n-dimensional Euclidean space. This is a vector space over $\mathbb{R}$. For example, we might consider the real line $\mathbb{R}$, the real plane $\mathbb{R}^2$ or $\mathbb{R}^3$. Fix a hyperplane P in V, i.e., a subspace of V of dimension $n-1$. In the real line $\mathbb{R}$ there is only one hyperplane:

In $\mathbb{R}^2$, they are the lines through the origin:



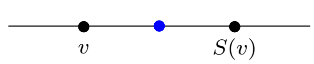
And in $\mathbb{R}^3$, they are the planes through the origin:



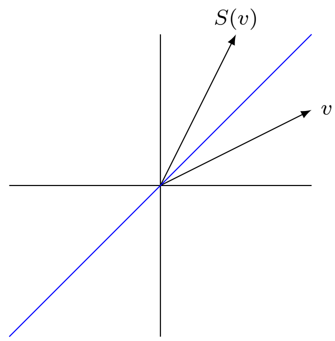
The reflection with respect to P is the function

$$S : V \to V$$

which sends each vector to its mirror image with respect to P. So, for example, in $\mathbb{R}$, there is only one reflection



In $\mathbb{R}^2$ we have:



What does it mean, exactly, to send a vector to its mirror image with respect to P? To define this precisely we need to recall that V is equipt with an inner product defined by

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

As usual, the length of a vector $x \in V$ is defined to be

$$\|x\| = (x, x)^{1/2}.$$

The Cauchy-Schwartz inequality asserts that

$$|(x, y)| \leq \|x\|\|y\|$$

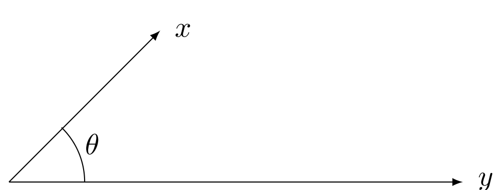
for $x, y \in V$. It follows that if $x, y \in V$ are nonzero, then

$$-1 \leq \frac{(x, y)}{\|x\|\|y\|} \leq 1.$$

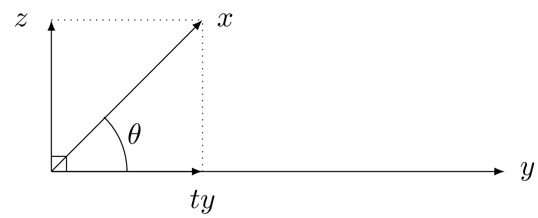
If $x, y \in V$ are nonzero, then we define the **angle** between x and y is defined to be the unique number $0 \leq \theta \leq \pi$ such that

$$(x, y) = \|x\|\|y\| \cos\theta.$$

The inner product measures the angle between two vectors, though it is a bit more complicated in that the lengths of x and y are also involved. The term "angle" does make sense geometrically. For example, suppose that $V = \mathbb{R}^2$ and we have:



Project x onto y, to obtain ty:



Then we have

$$x = z + ty.$$

Taking the inner product with y, we get

$$\begin{aligned}
(x, y) &= (z, y) + (ty, y) \\
(x, y) &= 0 + t(y, y) \\
(x, y) &= t\|y\|^2 \\
t &= \frac{(x, y)}{\|y\|^2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\cos\theta &= \frac{\|ty\|}{\|x\|} \\
\cos\theta &= t\frac{\|y\|}{\|x\|} \\
t &= \frac{\|x\|}{\|y\|}\cos\theta.
\end{aligned}$$

If we equate the two formulas for t we get $(x, y) = \|x\|\|y\|\cos\theta$. We say that two vectors are **orthogonal** if $(x, y) = 0$; if x and y are nonzero, this is equivalent to the angle between x and y being $\pi/2$. If $(x, y) > 0$, then we will say that x and y form an **acute** angle; this is equivalent to $0 < \theta < \pi/2$. If $(x, y) < 0$, then we will say that x and y form an **obtuse** angle; this is equivalent to $\pi/2 < \theta \leq \pi$. If X is a subset of V, then we define the orthogonal complement $X^\perp$ of X to be the set of all $y \in V$ such that $(x, y) = 0$ for all $x \in X$. The set $X^\perp$ is a subspace of V, even if X is not. If W is a subspace of V, then one has

$$V = W \oplus W^\perp.$$

With these definitions we can give a formal definition of a reflection with respect to a hyperplane P. From above, we have a decomposition

$$V = P \oplus P^\perp.$$

Since P is $n-1$ dimensional, $P^\perp$ is one dimensional, and spanned by a single vector. Thus, to define a linear transformation on V, it suffices to define it on P and $P^\perp$. We define the **reflection with respect to** P to be the the linear transformation $S : V \to V$ such that $Sx = x$ for x in P and $Sx = -x$ for $x \in P^\perp$. It is easy to see that

$$S^2 = 1;$$

a reflection has order two in the group of all invertible linear transformations. One can give a formula for S. Suppose that

$$P^\perp = \mathbb{R}r.$$

Then

Proposition 2.1. *We have*

$$Sx = x - 2\frac{(x,r)}{(r,r)}r.$$

Proof. Let T be the linear transformation defined by the above formula. If $x \in P$, then clearly $Tx = x$. If $x \in P^\perp$, say $x = cr$ for some $c \in \mathbb{R}$, then

$$\begin{aligned} Tx &= x - 2\frac{(cr,r)}{(r,r)}r \\ &= x - 2cr \\ &= x - 2x \\ &= -x. \end{aligned}$$

It follows that $T = S$. $\square$

Non-zero vectors also define some useful geometric objects. Let $r \in V$ be non-zero. We may consider three sets that partition V:

$$\{x \in V : (x,r) > 0\}, \qquad P = \{x \in V : (x,r) = 0\}, \qquad \{y \in V : (x,r) < 0\}.$$

The first set consists of the vectors that form an acute angle with r, the middle set is the hyperplane P orthogonal to $\mathbb{R}r$, and the last set consists of the vectors that form an obtuse angle with r. We refer to the first and last sets as the **half-spaces** defined by P. Of course, r lies in the first half-space. Let S be the reflection with respect to P. Using the formula from Proposition 2.1 shows that

$$(Sx,r) = -(x,r)$$

for x in V, so that S sends one half-space into the other half-space. Also, S acts by the identity on P. Multiplication by -1 also sends one half-space into the other half-space; however, while multiplication by -1 preserves P, it is not the identity on P.

3 The orthogonal group

We will be interested in the finite groups generated by reflections contained in the group of all invertible linear tranformations. As it turns out every reflection is actually contained inside a smaller group, called the orthogonal group. The **orthogonal group** $\mathrm{O}(V)$ is the set of all linear transformations T of V which preserve the angles between vectors, or equivalently, all linear transformations T such that

$$(Tx, Ty) = (x, y)$$

for $x, y \in V$. It is easy to see every element of the orthogonal group $\mathrm{O}(V)$ is indeed an invertible linear transformation, and that $\mathrm{O}(V)$ is a subgroup of the group $\mathrm{GL}(V)$ of all invertible linear transformations on V. Every reflection is contained in $\mathrm{O}(V)$:

Proposition 3.1. *If S is a reflection with respect to the hyperplane P, then S is contained in $\mathrm{O}(V)$.*

Proof. Using notation from above, we have for $x \in V$:

$$\begin{aligned}
(Sx, Sx) &= (x - 2\frac{(x,r)}{(r,r)}r, x - 2\frac{(x,r)}{(r,r)}r) \\
&= (x,x) - 2\frac{(x,r)}{(r,r)}(x,r) - 2\frac{(x,r)}{(r,r)}(r,x) + 4\frac{(x,r)^2}{(r,r)^2}(r,r) \\
&= (x,x) - 4\frac{(x,r)^2}{(r,r)} + 4\frac{(x,r)^2}{(r,r)} \\
&= (x,x).
\end{aligned}$$

This proves the proposition. □

In a moment we will compute the finite subgroups of $\mathrm{O}(V)$ when V is one, two and three dimensional. First, however, we will record a couple more general facts.

Proposition 3.2. *If $T \in \mathrm{O}(V)$ then $\det T = \pm 1$.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis for V, regarded as column vectors. We will prove this by computing (Te_i, Te_j) in two different ways. First, we have

$$(Te_i, Te_j) = (e_i, e_j) = \delta_{ij}.$$

On the other hand,

$$(Te_i, Te_j) = {}^t(Te_i)(Te_j) = {}^te_i{}^tTTe_j.$$

We thus have

$${}^te_i{}^tTTe_j = \delta_{ij}.$$

This is also the ij-th entry of ${}^tTT$. Hence, ${}^tTT = 1$, and so $\det T = \pm 1$. □

What is the determinant of a reflection?

Proposition 3.3. *If S is the reflection with respect to the hyperplane P, then* $\det S = -1$.

Proof. Choose a basis $v_1, \ldots, v_{n-1}$ for P. Then $r, v_1, \ldots, v_{n-1}$ is a basis for V. The matrix of S with respect to this basis is:

$$\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

The determinant of this matrix is -1. $\square$

We will say that $T \in \mathrm{O}(V)$ is a **rotation** if $\det T = 1$. This name for elements of $\mathrm{O}(V)$ with determinant one will be justified by proving that in the cases $\dim V = 2$ and $\dim V = 3$ a rotation is a rotation in the everyday sense. The subgroup of rotations in $\mathrm{O}(V)$ is often denoted by $\mathrm{SO}(V)$.

Reflections are not rotations, but a product of an even number of reflections is a rotation. If G is a subgroup of $\mathrm{O}(V)$, then the set of rotations in G forms a normal subgroup of index at most two:

Proposition 3.4. *Let G be a subgroup of* $\mathrm{O}(V)$, *and let H be the subset of all elements T G such that* $\det T = 1$. *Then H is normal in G, and* $[G : H] \leq 2$.

Proof. The map $\det : G \to \{\pm 1\}$ is a homomorphism with kernel H. Hence, H is normal in G, and G/H embeds in $\{\pm 1\}$, proving the result. $\square$

4 Finite subgroups in two dimensions

We will now determine the finite subgroups of $\mathrm{O}(V)$ when $n = 1, 2$ and 3. The situation when $n = 1$ is rather simple. Suppose that $T : \mathbb{R} \to \mathbb{R}$ is in $\mathrm{O}(V)$. Since V is one dimensional, it is spanned by a single vector, e.g., 1. With respect to this basis, T is given by multiplication by a number, say c. Since $(Tx, Ty) = (x, y)$ for all $x, y \in \mathbb{R}$, we have $c^2xy = xy$ for all $x, y \in \mathbb{R}$. This means $c = \pm 1$. The maps given by multiplication by ± 1 are both in $\mathrm{O}(V)$, and so $\mathrm{O}(V) = \{\pm 1\}$. The map -1 is a reflection.

Turning to the case of $\dim V = 2$, we have the following theorem.

Theorem 4.1. *Assume that* $\dim V = 2$*, and let* $T \in \mathrm{O}(V)$*. Then* T *is either a rotation or a reflection. If* T *is a rotation, then there exists* $0 \le \theta < 2\pi$ *such that the matrix of* T *is*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with respect to the standard basis for V*, and* T *rotates vectors in the counterclockwise direction through the angle* θ*. If* T *is a reflection, then there exists a* $0 \le \theta < 2\pi$ *such that the matrix of* T *is*

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

with respect to the standard basis for V*.*

Proof. Let's determine the matrix of an arbitrary element T of $\mathrm{O}(V)$ with respect to the standard basis. Suppose that

$$Te_1 = \mu e_1 + \nu e_2.$$

Since T preserves lengths, we must have

$$\begin{aligned} (Te_1, Te_1) &= (e_1, e_1) \\ (\mu e_1 + \nu e_2, \mu e_1 + \nu e_2) &= 1 \\ \mu^2(e_1, e_1) + \mu\nu(e_1, e_2) + \nu\mu(e_2, e_1) + \nu^2(e_2, e_2) &= 1 \\ \mu^2 + \nu^2 &= 1. \end{aligned}$$

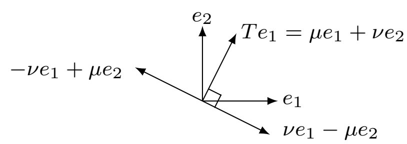
We also have

$$(Te_1, Te_2) = (e_1, e_2) = 0.$$

This means that, besides being of length one, Te_2 is orthogonal to Te_1. Since we are in two dimensional space, it is true that, and easy to prove that, there are only two vectors in V of length one which are orthogonal to Te_1. These are

$$-\nu e_1 + \mu e_2, \quad -(-\nu e_1 + \mu e_2).$$

The picture is:



We find that the matrix of T with respect to the standard basis is

$$\begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix}.$$

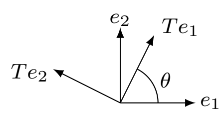
We can analyze this a bit more. As $\mu^2 + \nu^2 = 1$, there exists $0 \leq \theta < 2\pi$ such that

$$\mu = \cos\theta, \quad \nu = \sin\theta.$$

Then matrix of T is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Assume that the first possibility holds. Then evidently, as every vector is a linear combination of e_1 and e_2, and as T rotates e_1 and e_2 in the counterclockwise direction by θ, T is a rotation in the counterclockwise direction by θ:



It is also clear that $\det T = 1$ in this case; T is a rotation as defined before.

Next, suppose the second possibility holds. Squaring the matrix for T we find that $T^2 = 1$. Since reflections are of order two, perhaps T is a reflection. This is indeed true. One way to see this is to note that if

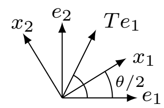
$$x_1 = \cos\frac{\theta}{2}e_1 + \sin\frac{\theta}{2}e_2 \quad x_2 = -\sin\frac{\theta}{2}e_1 + \cos\frac{\theta}{2}e_2$$

then direct computations show that

$$Tx_1 = x_1, \quad Tx_2 = -x_2.$$

As $(x_1, x_2) = 0$, T is the reflection with respect to the line spanned by x_1.

The picture is:



This proves the theorem. □

We can also interpret the theorem in terms of complex numbers. Assume that the notation is as in the statement of the last theorem, and regard V is regarded as $\mathbb{C}$. If T is a rotation with angle θ, then T is multiplication by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

If T is a reflection as in the theorem, then the matrix of T can also be written as

$$T = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

The matrix on the left is a rotation by θ, i.e., multiplication by $e^{i\theta}$, and the matrix on the right is the reflection through the x-axis, i.e., complex conjugation.

Corollary 4.2. *Assume that* $\dim V = 2$. *Then the subgroup of rotations in* $\mathrm{O}(V)$ *is abelian.*

Proof. By Theorem 4.1 the matrix of every rotation with respect to the standard basis has the form

$$\begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}$$

for some $\mu, \nu \in \mathbb{R}$. A calculation shows that matrices of this form commute with each other. □

Corollary 4.3. *Assume that* $\dim V = 2$. *Any two reflections are conjugate to each other via a rotation.*

Proof. We will write elements of $\mathrm{O}(V)$ in the standard basis. Let S be a reflection, so that

$$S = \begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix}$$

for some real numbers μ and ν such that $\mu^2 + \nu^2 = 1$. To prove the corollary, it will suffice to prove that there exist real numbers a and b, not both zero, such that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = \begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

This equality holds if and only if

$$\begin{bmatrix} 1-\mu & -\nu \\ \nu & -(1+\mu) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since

$$\det \begin{bmatrix} 1-\mu & -\nu \\ \nu & -(1+\mu) \end{bmatrix} = 0,$$

this system has a non-zero solution. □

Next, we turn to the problem of finding all the finite subgroups of $\mathrm{O}(V)$ when $n = 2$. What are some examples of finite subgroups of $\mathrm{O}(V)$? Suppose that n is a positive integer, and let $\theta = 2\pi/n$. Consider the element R of $\mathrm{O}(V)$ which is a rotation in the counterclockwise direction through the angle θ. The subgroup generated by R clearly is of order n, and the elements have matrices

$$R^k = \begin{bmatrix} \cos(k2\pi/n) & -\sin(k2\pi/n) \\ \sin(k2\pi/n) & \cos(k2\pi/n) \end{bmatrix}, \quad 0 \le k \le n-1$$

with respect to the standard basis. We will denote this subgroup by $\mathcal{C}_2^n$. This group, of course, does not contain any reflections.

Are there any more examples? Yes: we can enlarge $\mathcal{C}_2^n$ by using a reflection. Let S be any reflection. Let's consider the subgroup generated by R and S inside $\mathrm{O}(V)$. Then RS is also a reflection because $\det RS = -1$, and by the above theorem, RS must be a reflection. This implies $SRSR = 1$. This also can be verified geometrically. Hence, $RS = SR^{-1} = SR^{n-1}$. From this, we see that every element of the subgroup generated by R and S is in the following list:

$$1, R, \ldots, R^{n-1}, S, SR, \ldots, SR^{n-1}.$$

All the elements on this list are distinct, and so this is the entire group. This group is called the **dihedral group** of order $2n$, and is denoted by $\mathcal{H}_2^n$. We do not denote the dependence on S in the notation for the dihedral group. The reason is that these groups are all very similar, and in fact conjugate to each other inside $\mathrm{O}(V)$ by Corollary 4.2 and Corollary 4.3. The group $\mathcal{H}_2^n$ is generated by reflections, in contrast to $\mathcal{S}_2^n$. The following theorem proves that these are all the possible finite subgroups of $\mathrm{O}(V)$.

Theorem 4.4. *Let $V = \mathbb{R}^2$. Then the finite subgroups of $\mathrm{O}(V)$ are the groups $\mathcal{C}_2^n$ and $\mathcal{H}_2^n$ for n a positive integer.*

Proof. Let G be a finite subgroup of $\mathrm{O}(V)$. We have seen that the subgroup H of G of rotations is of index at most two. Let us first determine the structure of H. If $H = 1$, there is nothing more to say; assume $H \neq 1$. If $R \in H$, then we proved that R is a rotation through an angle θ in the counterclockwise direction with $0 \le \theta < 2\pi$. Let R be the nontrivial rotation with θ minimal; this exists, as H is finite. Next, for each $T \in H$, pick an integer m such that

$$m\theta \leq \theta(T) < (m+1)\theta.$$

Then

$$0 \leq \theta(T) - m\theta < \theta < 2\pi.$$

In fact, $\theta(T) - m\theta$ is the angle for a rotation in H, namely $R^{-m}T$: $R^{-m}T$ is a counter-clockwise rotation through the angle $\theta(T)$, followed by m clockwise rotations through the angle $\theta(R)$. This means that

$$\theta(R^{-m}T) = \theta(T) - m\theta.$$

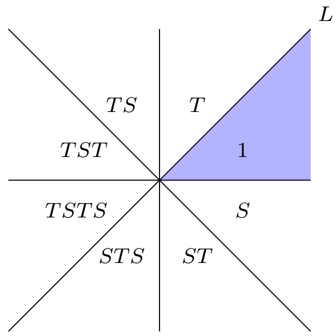
By the minimality of θ, this must be zero, which means

$$T = R^m.$$

We have proven that $H = \mathcal{C}_2^{|H|}$.

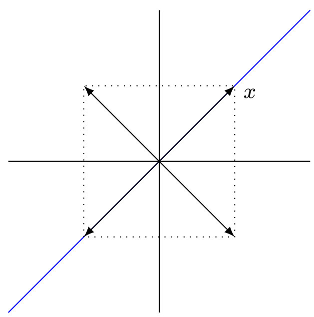
If $G = H$, we are done; assume $H \neq G$. Then H is a subgroup of G of index two. Let $S \in G$ with $S \notin H$. Then $\det S = -1$; by the above theorem, this implies S is a reflection. Since H has index two, G is generated by R and S. Therefore, $G = \mathcal{H}_2^n$ (for this choice of S). $\square$

We can draw some pictures concerning the dihedral group. Let n be a positive integer, and set $\theta = 2\pi/n$. As usual, let R be the rotation through θ degrees. Let S be the reflection through the x-axis. We consider the dihedral group $\mathcal{H}$ generated by R and S. This is also generated by $T = RS$ and S. What is T? It is a reflection, because it has determinant -1, and in two dimensions, an element of $\mathrm{O}(V)$ with determinant -1 is a reflection. What line is it a reflection through? It is the line through the origin which makes an angle $\theta/2$ with the x-axis. Let us call this line L. Then a useful picture associated with $\mathcal{H}$ can be drawn as follows. Let F be the open region between the x-axis and L. It is not too hard to see that no two points of F can be mapped to each other using a nonidentity element of $\mathcal{F}$ (certainly, any nonidentity power of R cannot map two points of F to each other; the same is true for all elements of the form R^kS for $0 \leq k \leq n-1$). Thus, if we apply the elements of $\mathcal{H}$ to F, we will obtain as many open regions as there are elements of $\mathcal{H}$, namely $2n$. Each of these will be one of other $2n-1$ wedges through an angle $\theta/2$ (applying powers of R to the wedge F gives n such regions; applying powers of R to SF gives the rest. We can label each of the wedges with the element which yields this wedge when applied to F. It gives a kind of picture of $\mathcal{H}$. This is illustrated in the case $n = 4$:



We call F a fundamental region for $\mathcal{F}$; we will discuss fundamental regions for reflection groups later on.

There is another picture which also helps in understanding $\mathcal{H}$. Let x be the vector of length one on the line L pointing in the positive direction. If we apply powers of R, we obtain n vectors of the same length, with an angle $2\pi/n$ between each vector. Connect the vectors with line segments. The result is a regular n-gon X. The group $\mathcal{H}$ is the subgroup of elements of $\mathrm{O}(V)$ which map X to itself: it is the symmetry group of X (certainly, the symmetry group of X is a finite subgroup of $\mathrm{O}(V)$ containing $\mathcal{H}$. But the symmetry group can be no bigger by the classification of finite subgroups of $\mathrm{O}(V)$) . In the case $n = 4$ we have:



5 Finite subgroups in three dimensions

As in the case of two dimensions, we will determine the finite subgroups of $\mathrm{O}(V)$. Just as in the the two dimensional case, we first need some general structure theorems about the elements of $\mathrm{O}(V)$.

Lemma 5.1. *Suppose that V has dimension n (not necessarily three), and $T \in \mathrm{O}(V)$. If λ is an eigenvalue of T, then $\lambda\bar{\lambda} = 1$.*

Proof. We regard $\mathbb{R}^n$ as contained in $\mathbb{C}^n$. We can extend the inner product on $\mathbb{R}^n$ to an inner product on $\mathbb{C}^n$ by

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i.$$

The map $T : \mathbb{R}^n \to \mathbb{R}^n$ extends to a map $\mathbb{C}^n \to \mathbb{C}^n$ by linearity, and we have again $(Tx, Ty) = (x, y)$ for $x, y \in \mathbb{C}^n$. There exists a nonzero vector x in $\mathbb{C}^n$ such that $Tx = \lambda x$. We get

$$(x, x) = (Tx, Tx) = \lambda\bar{\lambda}(x, x).$$

Since $(x, x) \neq 0$, we get $\lambda\bar{\lambda} = 1$. $\square$

Theorem 5.2. (Euler) *Assume that* $\dim V = 3$. *Let $T \in \mathrm{O}(V)$ be a rotation. Then T is a rotation about a fixed axis, in the sense that T has a eigenvector x with eigenvalue* 1 *such that the restriction to $P = x^{\perp}$ is a two dimensional rotation of P.*

Proof. Consider the characteristic polynomial of T; let λ_1, λ_2 and λ_3 be its roots in $\mathbb{C}$. If λ is a root of the characteristic polynomial, then $\bar{\lambda}$ is also a root. Since the characteristic polynomial of T is of degree three, one of the roots, say λ_1, of the characteristic polynomial is real. By the lemma, $\lambda_1 = \pm 1$. In addition, we have $\det T = \lambda_1\lambda_2\lambda_3 = 1$. Hence, $\lambda_2\lambda_3$ is also real. If λ_2 is not real, then $\bar{\lambda}_2 = \lambda_3$, and $\lambda_2\lambda_3 = \lambda_2\bar{\lambda}_2 = 1$, so that $\lambda_1 = 1$. If λ_2 is real, then so is λ_3; by the lemma we have $\lambda_2 = \pm 1$ and $\lambda_3 = \pm 1$; since $\lambda_1\lambda_2\lambda_3 = 1$, at least one eigenvalue is 1. We thus, in any case, may assume $\lambda_1 = 1$. Let x be an eigenvector for the eigenvalue 1. Consider $P = x^{\perp}$; we claim that T preserves P. For let $y \in P$. Then

$$(Ty, x) = (y, T^{-1}x) = (y, x) = 0.$$

Thus, $TP = P$. Consider now the restriction of T to P. Since $Tx = x$, and $\det T = 1$, we must have $\det(T|_P) = 1$. That is, the restriction of T to P is a rotation. $\square$

The previous theorem asserts that in three dimensions, rotations really are rotations in the everyday sense.

In the two dimensional case it was also important to understand the elements of $\mathrm{O}(V)$ with determinant -1: what can be said in the three dimensional case about such elements?

Theorem 5.3. *Suppose $T \in \mathrm{O}(V)$ has $\det T = -1$. Then T is a reflection through a plane P, followed by a rotation about the line through the origin orthogonal to P.*

Proof. Arguing just as in the proof of the last theorem, -1 is an eigenvalue for T. Let x be an eigenvector of eigenvalue -1. Again, let $P = x^{\perp}$. Just as before, $TP = P$; and now $\det T|_P = 1$. This means that $T|_P$ is a rotation. We can choose an orthonormal basis x_2, x_3 for P such that the matrix for $T|_P$ is:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some $0 \leq \theta < 2\pi$. The matrix of T in the basis x, x_2, x_3 is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

We can write this as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix on the right is that of the reflection through P; it is followed by the rotation about the line through the origin orthogonal to P through an angle θ. □

We will start our classification of the finite subgroups of $\mathrm{O}(V)$ when V is three dimensional by first determining finite subgroups consisting of rotations. It will be easy to obtain all finite subgroups of $\mathrm{O}(V)$ from a list of all finite rotation subgroups since 3 is odd.

To start, we consider the rotation subgroups that come from two dimensions. Let $V = \mathbb{R}^3$, and inside V pick a plane W through the origin; we call it W instead of P because we want to distinguish it from the hyperplane determining a reflection. It turns out that there is a way to extend any element of $\mathrm{O}(W)$ to a rotation of V. Define

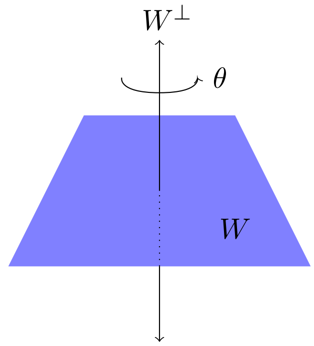
$$\mathrm{O}(W) \hookrightarrow \mathrm{SO}(V)$$

by sending $T \in \mathrm{O}(W)$ to the map $T : V \to V$ defined by letting T be defined as before on W, and by letting $Tx = \det T \cdot x$ for $x \in W^{\perp}$. In terms of matrices, this map is given by

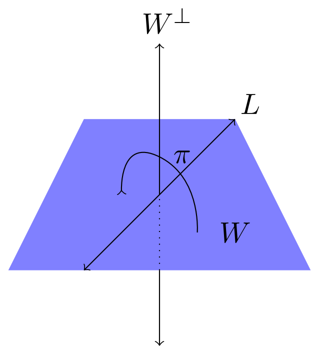
$$T \mapsto \begin{bmatrix} \det T & 0 \\ 0 & T \end{bmatrix}.$$

It is clear that this map is a homomorphism, and thus gives an injection of $\mathrm{O}(W)$ into $\mathrm{SO}(V)$.

What finite rotation subgroups do we obtain by using this map? Before, we saw that the finite subgroups of rotations in two dimensions were the cyclic subgroups $\mathcal{C}_2^n$ for positive integers n. Using the above inclusion, we obtain finite cyclic subgroups of rotations in $\mathrm{O}(V)$ which we will denote by $\mathcal{C}_3^n$. In terms of pictures, we have:



In addition to finite cyclic groups, $\mathrm{O}(W)$ also contains the finite dihedral groups. We will denote the image of $\mathcal{H}_2^n$ by $\mathcal{H}_3^n$. It is natural to wonder what happens to the element S of $\mathcal{H}_2^n$ under the inclusion. In $\mathrm{O}(W)$ it is a reflection, but it maps to a rotation in three dimensions. It is the rotation through an angle π around the line L. The picture is:



How many different finite rotation subgroups of $\mathrm{O}(V)$ do we obtain in this way? Based on group structure and orders, the only possible pair of subgroups that could be isomorphic are $\mathcal{S}_3^2$ and $\mathcal{H}_3^1$, which both have order two. Each of these consists of the identity transformation along with another element which is a rotation through π degrees.

What about other finite rotation groups? We saw in two dimensions that the finite subgroups of $\mathrm{SO}(W)$ and $\mathrm{O}(W)$ arise as symmetry groups of the regular n-gons. It is thus natural to consider the same kind of source in three dimensions. There are five regular convex polyhedra in three dimensions: the tetrahedron, the octahedron, the cube, the dodecahedron, and the icosahedron. We regard this as being centered at the origin in $\mathbb{R}^3$. We ask: what are the rotations which preserve these polyhedra, i.e., what are the rotational symmetry groups of these solids? In fact, the octahedron and the cube have the same rotational symmetry group, and the dodecahedron and the icosahedron have the same rotation symmetry group. The reason is because these members of these pairs of solids are dual to each other: if in one member one connects the center points of the faces, then one gets the other member of the pair. Thus, we need to only consider the tetrahedron, the cube, and the icosahedron.

We will denote the rotational symmetry group of the tetrahedron by $\mathcal{T}$. The elements of $\mathcal{T}$ can be listed as follows. If we draw a line from a vertex through the center point of the opposite face, then there are two rotations about this axis which preserve the tetrahedron, through angles $2\pi/3$ and $2 \cdot 2\pi/3$. Next, we can pick the midpoint of an edge, and draw the line through the midpoint of the opposite edge: there is a rotation through an angle π through this axis which preserves the tetrahedron. Of course, $\mathcal{T}$ also contains the identity transformation. So we obtain:

$$|\mathcal{T}| = 4 \cdot 2 + 3 \cdot 1 + 1 = 12.$$

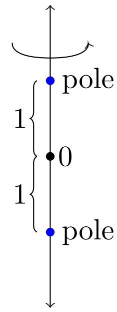
Next, let $\mathcal{W}$ be the rotational symmetry group of the cube. Besides the identity element, $\mathcal{W}$ consists of: the rotations around the axes through the center points of opposite faces with angles $2\pi/4, 2 \cdot 2\pi/4$ and $3 \cdot 2\pi/4$; the rotations around the axes through opposing vertices with angles $2\pi/3$ and $2 \cdot 2\pi/3$; and the rotations around the axes through the center points of opposing edges with angle $2\pi/2$. Hence,

$$|\mathcal{W}| = 3 \cdot 3 + 4 \cdot 2 + 6 \cdot 1 + 1 = 24.$$

Finally, let $\mathcal{I}$ be the rotational symmetry group of the icosahedron. It has 20 faces, 30 edges, and 12 vertices. Besides the identity element, $\mathcal{I}$ consists of: the rotations about the axes through the center points of opposing faces with angles $2\pi/3$ and $2 \cdot 2\pi/3$; the rotations about the axes through opposing vertices with angles $2\pi/5, 2 \cdot 2\pi/5, 3 \cdot 2\pi/5$ and $4 \cdot 2\pi/5$; the rotations though the center points of opposing edges with angle $2\pi/2$. Hence,

$$|\mathcal{I}| = 10 \cdot 2 + 6 \cdot 4 + 15 \cdot 1 + 1 = 60.$$

So far, we proved some general structural theorems about elements of $\mathrm{O}(V)$, and found some examples of finite rotation groups in $\mathrm{O}(V)$ when V is three dimensional. Next, we will prove that we have in fact found all finite rotation groups. To do this, we need to introduce another concept. Suppose that T in $\mathrm{O}(V)$ is a rotation and $T \neq 1$, with V three dimensional. Because T is a member of the orthogonal group it preserves length, and thus permutes the points of the unit ball, i.e., all the vectors of length one. But more is true: by Euler's theorem, since T is just a rotation about some axis, T fixes exactly two points on the unit sphere, namely the two points where the axis of rotation intersects the unit sphere. We will call these two points the **poles** of T. The picture is:



Lemma 5.4. *Assume V is three dimensional, and G is a group of rotations of V, i.e., a subgroup of* $\mathrm{SO}(V)$. *Let $\mathcal{S}$ be the set of all the poles of G. Then G permutes $\mathcal{S}$.*

Proof. Let $x \in \mathcal{S}$, and $R \in G$. We need to show that $Rx \in \mathcal{S}$. Since $x \in \mathcal{S}$, there exists $T \in G$ such that $Tx = x$. We have

$$(RTR^{-1})Rx = RTx = Rx.$$

As $RTR^{-1} \in G$, we have $Rx \in \mathcal{S}$. $\square$

This lemma can be used as a basis for obtaining a condition on G which will lead to a determination of all the finite rotation groups in three dimensions. First, however, we look at some examples:

Proposition 5.5. *We have*

G	$\lvert G \rvert$	*number of poles* $= \lvert\mathcal{S}\rvert$	*Orbits*	*Orders of orbits*	*Orders of stabilizers*
$\mathcal{C}_3^n$	n	2	2	$1, 1$	n, n
$\mathcal{H}_3^n$	$2n$	$2n+2$	3	$n, n, 2$	$2, 2, n$
$\mathcal{T}$	12	14	3	$6, 4, 4$	$2, 3, 3$
$\mathcal{W}$	24	26	3	$12, 8, 6$	$2, 3, 4$
$\mathcal{I}$	60	62	3	$30, 20, 12$	$2, 3, 5$

Proof. $\mathcal{C}_3^n$: This is the group of rotations generated by a single rotation about an axis through angle $2\pi/n$. The points on the unit sphere fixed by these rotations are the points of distance 1 on the axis from the origin, and there are two such points. Hence, $|\mathcal{S}| = 2$. These points clear like in different orbits, and both are stabilized by every point of $\mathcal{C}_3^n$.

$\mathcal{H}_3^n$: This is the rotational symmetry group of the regular n-gon. The $2n-1$ nontrivial rotations in this group are divided into two sets. In the first set are the $n-1$ non-trivial rotations with common axis through the center of the n-gon perpendicular to the n-gon; these all share the same 2 poles. In the second set are the n non-trivial rotations with axes in the same plane as the n-gon; the vertices and the center points of edges of the n-gon are the poles of these rotations, and there are $2n$ such poles. Altogether there are $2n+2$

poles. The two poles which lie on the axis through the center of the n-gon clearly form an orbit, and the stabilizer of an element of this orbit has order n. The vertices of n-gon form another orbit with n elements, and the stabilizer of an element of this orbit has order 2. The midpoints of the sides of the n-gon form another orbit with n elements, and the stabilizer of an element of this orbit has order 2.

$\mathcal{T}$: The nontrivial rotations in this group have together 7 axes, and each axes has two poles, so that there are 14 poles. The vertices of $\mathcal{T}$ all lie in the same orbit, and they form an orbit with four elements. The order of a stabilizer of an element in this orbit is 3. Similar comments apply to the midpoints of the faces, which may also be regarded as poles. Finally, the midpoints of edges also form an orbit with 6 elements, and the stabilizer of a point in this orbit has 2 elements.

$\mathcal{W}$: The nontrivial rotations in this group have together 13 axes, and each axes has two poles, so there are 26 poles. The center points of opposing edges are all poles, and form an orbit. There are 12 such points, and the order of a stabilizer of a such a point is 2. The vertices of the cube all lie in the same orbit, and they form an orbit with 8 elements; the order of a stabilizer of an element in this orbit is 3. The center points of opposite faces are all poles, and form an orbit. There are 6 such points, and the order of a stabilizer of such a point is 4.

$\mathcal{I}$: The nontrivial rotations in this group have together 31 axes, and each axes has two poles so there are 62 poles. The remaining analysis is similar to the last two cases. □

Theorem 5.6. *Let V be three dimensional, and let G be a finite group of rotations of V. Consider the action of G on its set $\mathcal{S}$ of poles, so that there is a partition into orbits:*

$$\mathcal{S} = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k.$$

Let $n = |G|$ and $v_i = |\mathcal{O}_i|$. Then

$$2 - \frac{2}{n} = \sum_{i=1}^{k} 1 - \frac{v_i}{n}.$$

Proof. Let $\mathcal{U}$ be the set of all pairs (T, x) where $T \in G$ is a nonidentity element and x is a pole of T. We will count $\mathcal{U}$ in two different ways. First, based on counting starting from a nonidentity group element, as each such element has exactly two poles, we have

$$|\mathcal{U}| = 2(n-1).$$

Second, we can count by starting from a pole. Fix $x \in S$. The map

$$\{(T, y) \in \mathcal{U} : y = x\} \longrightarrow G_x - \{1\}$$

defined by $T \mapsto T$ is clearly a bijection. Hence,

$$|\mathcal{U}| = \sum_{x \in \mathcal{S}} (|G_x| - 1).$$

We can further compute this sum. For each i, fix an element $x_i \in \mathcal{O}_i$. Then:

$$\begin{aligned}
\sum_{x\in\mathcal{S}}(|G_x|-1) &= \sum_{i=1}^{k}\sum_{x\in\mathcal{O}_i}(|G_x|-1)\\
&= \sum_{i=1}^{k}\sum_{x\in\mathcal{O}_i}(|G_{x_i}|-1)\\
&= \sum_{i=1}^{k}(|G_{x_i}|-1)\sum_{x\in\mathcal{O}_i}1\\
&= \sum_{i=1}^{k}(|G_{x_i}|-1)|\mathcal{O}_i|\\
&= \sum_{i=1}^{k}|G_{x_i}||\mathcal{O}_i|-|\mathcal{O}_i|\\
&= \sum_{i=1}^{k}n-v_i.
\end{aligned}$$

Equating the two ways of counting $|\mathcal{U}|$ and dividing by n gives the result. □

Theorem 5.7. *If G is a finite rotation group, then G is conjugate in $\mathrm{O}(V)$ to $\mathcal{C}_3^n$, $n \geq 1$, $\mathcal{H}_3^n$, $n \geq 2$, $\mathcal{T}$, $\mathcal{W}$ or $\mathcal{I}$.*

Proof. Let G be a finite rotation group. We will first show that $|G|$, $|\mathcal{S}|$, the number of orbits and their size must be as on one of the lines of the above table. To prove this, we first note that we may assume $n > 1$. This implies:

$$1 \leq 2 - \frac{2}{n} < 2.$$

Now n/v_i is the number of elements in the stabilizer of any element of $\mathcal{O}_i$, and every pole is stabilized by at least two elements: hence,

$$n/v_i \geq 2.$$

This implies

$$\begin{aligned}
v_i/n &\leq 1/2\\
-v_i/n &\geq -1/2\\
1-v_i/n &\geq 1/2,
\end{aligned}$$

so that

$$\frac{1}{2} \leq 1 - \frac{v_i}{n} < 1.$$

Since

$$2 - \frac{2}{n} = \sum_{i=1}^{k} 1 - \frac{v_i}{n}$$

we conclude that $k = 2$ or $k = 3$.

Assume $k = 2$. Then

$$\begin{aligned} 2 - \frac{2}{n} &= (1 - \frac{v_1}{n}) + (1 - \frac{v_2}{n}) \\ \frac{2}{n} &= \frac{v_1}{n} + \frac{v_2}{n} \\ 2 &= v_1 + v_2. \end{aligned}$$

We must have $v_1 = v_2 = 1$. This gives the $\mathcal{C}_3^n$ line of the table.

Assume $k = 3$. We may assume $v_1 \geq v_2 \geq v_3$. Then

$$\frac{1}{2} \leq 1 - \frac{v_1}{n} \leq 1 - \frac{v_2}{n} \leq 1 - \frac{v_3}{n}.$$

Also, we have

$$(1 - \frac{v_1}{n}) + (1 - \frac{v_2}{n}) + (1 - \frac{v_3}{n}) = 2 - \frac{2}{n} < 2.$$

This means

$$\begin{aligned} 1 - \frac{v_1}{n} &< \frac{2}{3} \\ \frac{v_1}{n} &> \frac{1}{3} \\ \frac{n}{v_1} &< 3. \end{aligned}$$

The number n/v_1 is a positive integer greater than or equal to 2 (it is the order of the stabilizer of any point in $\mathcal{O}_1$) and therefore

$$\frac{n}{v_1} = 2.$$

That is,

$$v_1 = \frac{n}{2}.$$

This tells us that

$$\begin{aligned} (1 - \frac{n/2}{n}) + (1 - \frac{v_2}{n}) + (1 - \frac{v_3}{n}) &= 2 - \frac{2}{n} < 2, \\ \frac{1}{2} + (1 - \frac{v_2}{n}) + (1 - \frac{v_3}{n}) &= 2 - \frac{2}{n} < 2, \end{aligned}$$

$$(1-\frac{v_2}{n})+(1-\frac{v_3}{n}) \quad = \quad \frac{3}{2}-\frac{2}{n}<\frac{3}{2}.$$

We must have

$$\begin{aligned}
1-\frac{v_2}{n} &< \frac{3}{2\cdot 2}\\
-\frac{v_2}{n} &< -\frac{1}{4}\\
\frac{v_2}{n} &> \frac{1}{4}\\
\frac{n}{v_2} &< 4.
\end{aligned}$$

Again, n/v_2 is an positive integer which is at least two. This means that

$$v_2=\frac{n}{2} \quad \text{or} \quad v_2=\frac{n}{3}.$$

Assume $v_2=n/2$. Then

$$\begin{aligned}
(1-\frac{n/2}{n})+(1-\frac{v_3}{n}) &= \frac{3}{2}-\frac{2}{n}\\
1-\frac{v_3}{n} &= 1-\frac{2}{n}\\
v_3 &= 2.
\end{aligned}$$

That is, in this case we have:

$$v_1=\frac{n}{2}, v_2=\frac{n}{2}, v_3=2.$$

This is the $\mathcal{H}_3^n$ line of the table.

Finally, assume $v_2=n/3$. Then computations show that

$$\frac{1}{n/v_3}=\frac{1}{6}+\frac{2}{n}.$$

If $n/v_3=1$ or $n/v_3=2$, then $v_2=n/3<v_3$, a contradiction. If $n/v_3=3$ then $n=12$, and $v_1=6$, $v_2=4$ and $v_3=4$. This is the $\mathcal{T}$ line of the table. If $n/v_3=4$, then $n=24$, $v_1=12$, $v_2=8$ and $v_3=6$. This the $\mathcal{W}$ line of the table. If $n/v_3=5$, then $n=60$, $v_1=30$, $v_2=20$ and $v_3=12$. This is the $\mathcal{I}$ line of the table. Finally, $n/v_3\geq 6$ is impossible as it would imply $2/n<0$.

We have proven that G satisfies the conditions in the last five entries of one of the rows of the table in the statement of Propostion 5.5; next, we will prove that G is in fact conjugate in $\mathrm{O}(V)$ to one of the groups listed in the statement of the theorem we are proving.

Suppose first that G satisfies the conditions in the last five entries of the first row of the table. Since the non-identity elements of G have the same two poles, it follows from Theorem 5.2 that the elements have a common eigenvector x and the restrictions of the elements of G to $P = x^{\perp}$ are rotations. By the proof of Theorem 4.4, the group of restrictions of the elements of G to P is $\mathcal{C}_2^n$. It follows that G is conjugate in $\mathrm{O}(V)$ to $\mathcal{C}_3^n$.

Suppose that G satisfies the conditions in the last five entries of the second row of the table and that $2n > 2$, so that $n \geq 2$. We need to prove that G is conjugate in $\mathrm{O}(V)$ to $\mathcal{H}_{2n}^3$. Let x be a pole that has an orbit of order 2, and let G_x be the stabilizer in G of x. Then the order of G_x is n. The elements of G_x map the space $P = x^{\perp}$ into itself, and the restrictions of the elements of G_x to $P = x^{\perp}$ are rotations. As in the last paragraph, the group of restrictions of the elements of G_x to P is $\mathcal{C}_2^n$ with respect to P.

Let y be a pole of G not in the orbit of x; we will prove that $y \in P$. We have $V = \mathbb{R}x \oplus P$. Write $y = cx + z$ for some $c \in \mathbb{R}$ and $z \in P$. We first claim that $z \neq 0$. For suppose that $z = 0$. Then $y = cx$, and $c = \pm 1$; since $x \neq y$, $c = -1$ so that $y = -x$. This implies that $G_x = G_y$. The index of $G_x = G_y$ in G is 2; let T be representative for the non-trivial coset of $G_x = G_y$ in G. The orbit of x is $\{x, Tx\}$ and the orbit of y is $\{y, Ty\}$. By the table, we must have $n = 2$, so that G has order 4, and $G_x = G_y$ has order 2. Since G has order 4, G is abelian. Let $G_x = G_y = \{1, R\}$. We have $Rx = x$, so that $R(Tx) = Tx$. This implies that Tx is a pole of R, and hence that $Tx = \pm x$. Since the orbit $\{x, Tx\}$ has order 2, we must have $Tx = -x = y$, contradicting the assumption that y is not in the orbit of x. Thus, $z \neq 0$. Next, let $S \in G_y$ be non-trivial. Since G_x has index two in G, G_x is normal in G, so that $S^{-1}G_xS = G_x$. Let R be a non-trivial element of G_x. Then $S^{-1}RSx = x$, or equivalently, $R(Sx) = Sx$. This implies that Sx is a pole of R; since the two poles of R are x and $-x$, we have $Sx = x$ or $Sx = -x$. Assume first that $Sx = x$, so that $S \in G_x$; we will obtain a contradiction. Since S is non-trivial and $Sx = x$, we must have that $S|_P$ is a non-trivial rotation of P; in particular, $Sz \neq z$ because $z \neq 0$. We now have

$$\begin{aligned} Sy &= cSx + Sz \\ y &= cx + Sz \\ cx + z &= cx + Sz \\ z &= Sz, \end{aligned}$$

which is a contradiction. Thus, $Sx = -x$. Calculating again, we have

$$\begin{aligned} Sy &= cSx + Sz \\ y &= -cx + Sz \\ cx + z &= -cx + Sz \\ 2cx &= -z + Sz. \end{aligned}$$

The vector $2cx$ lies in $\mathbb{R}x$ while $-z+Sz$ lies in P. Therefore, $2cx = -z+Sz = 0$, so that $c = 0$ and hence $y \in P$, as claimed.

Now we can complete the argument that G is conjugate to $\mathcal{H}_3^n$ inside $\mathrm{O}(V)$. By the table, G_y has order 2; this means that S has order 2 and is thus a rotation by π degrees about the line through y. Since the poles $\pm y$ of S lie in P by the last paragraph, it is now evident that G is conjugate to $\mathcal{H}_3^n$ inside $\mathrm{O}(V)$.

The remaining cases of the table will be omitted. □

We are close to being able to state the classification of finite subgroups of $\mathrm{O}(V)$ when V is three dimensional. This classification will use the classification of finite rotation subgroups. To use this classification we will need to describe two ways of constructing subgroups of $\mathrm{O}(V)$ from rotation subgroups. These two constructions depend on the fact that

$$-1 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \in \mathrm{O}(V), \quad \det(-1) = -1.$$

By -1 we mean multiplication of vectors by -1. This can be viewed as the reflection through the xy-plane, followed by a rotation through an angle π around the z-axis. The element -1 has the property that it lies in the center of $\mathrm{O}(V)$.

To describe the first construction, suppose that H is a group of rotations. Consider the set

$$H^* = H \cup (-1)H.$$

It is easy to see that this set forms a subgroup of $\mathrm{O}(V)$; moreover, $|H^*| = 2|H|$.

The second construction will only apply to certain rotation subgroups. Suppose that K is a group of rotations and K contains a subgroup H of index 2. Consider the set

$$K]H = H \cup \{-T : T \in K - H\}.$$

It can be verified that this is a subgroup of $\mathrm{O}(V)$. We have $|K]H| = |K| = 2|H|$.

Proposition 5.8. *Let G be a subgroup of $\mathrm{O}(V)$, and let H be the subgroup of G of rotations. Then exactly one of the following holds*

i) G is a group of rotations;

ii) G is not a group of rotations and $-1 \in G$, in which case $G = H^*$;

iii) G is not a group of rotations and $-1 \notin G$, in which case $G = K]H$ for some group of rotations K containing H as a subgroup of index two.

Proof. If $G = H$, then the first case holds. If $-1 \in G$, then clearly $G = H^*$. Assume $G \neq H$ and $-1 \notin G$. Let $S \in G$, $S \notin H$. Then $S^2 \in H$ and $\det S = -1$. Set

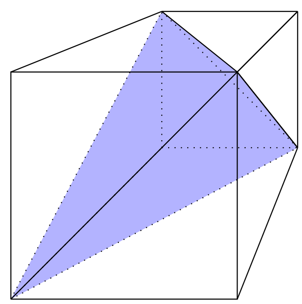
$$K = H \cup (-S)H.$$

Computations show that this is a group of rotations. It clearly contains H as a subgroup of index two because otherwise $-S \in H$, which implies that $-1 \in G$. We have

$$K]H = H \cup \{-T : T \in K - H\} = H \cup \{-(-S)R : R \in H\} = H \cup SH = G.$$

This completes the proof. $\square$

The following picture of a tetrahedron inside a cube shows that there is an embedding of $\mathcal{T}$ in $\mathcal{W}$ as a subgroup of index 2.



Theorem 5.9. *Every finite subgroup of* $\mathrm{O}(V)$ *is conjugate in* $\mathrm{O}(V)$ *to one of the following subgroups, and no two distinct subgroups of this list are conjugate:*

i) $\mathcal{C}_3^n$, $n \geq 1$, $\mathcal{H}_3^n$, $n \geq 2$, $\mathcal{T}$, $\mathcal{W}$, $\mathcal{I}$;

ii) $\mathcal{C}_3^{n*}$, $n \geq 1$, $\mathcal{H}_3^{n*}$, $n \geq 2$, $\mathcal{T}^*$, $\mathcal{W}^*$, $\mathcal{I}^*$;

iii) $\mathcal{C}_3^{2n}]\mathcal{C}_3^n$, $n \geq 1$, $\mathcal{H}_3^n]\mathcal{C}_3^n$, $\mathcal{H}_3^{2n}]\mathcal{H}_3^n$, $n \geq 2$, $\mathcal{W}]\mathcal{T}$.

Proof. One can prove the rotation groups which admit subgroups of index two are as listed in iii). By Proposition 5.8, it follows that every finite subgroup of $\mathrm{O}(V)$ is conjugate to one of the groups on the list. One can verify that no two groups on the list are conjugate. $\square$

6 Fundamental regions

Throughout this section $V = \mathbb{R}^n$.

We will now describe the Fricke-Klein construction of a fundamental region or domain for the action of a finite subgroup of $\mathrm{O}(V)$ on V for n-dimensional V. Let G be a finite subgroup of $\mathrm{O}(V)$. A subset $F \subset V$ is called a **fundamental region** for G in V if

i) F is open;

ii) $F \cap TF = \varnothing$ for $T \in G$, $T \neq 1$;

iii) $V = \cup_{T \in G} \overline{TF}$.

Sometimes it is also useful to consider the action of G on some subset X of G. Suppose $TX \subset X$ for $T \in G$. Then one can also define the concept of a fundamental region for G in X. A fundamental region $F \subset X$ is a relatively open subset such that $F \cap TF = \varnothing$ for $T \in G$, $T \neq 1$, and $X = \cup_{T \in G} X \cap \overline{TF}$.

Lemma 6.1. *For $n \geq 1$, the vector space V is not the union of a finite number of proper subspaces.*

Proof. We will prove this by induction on n. The proposition clearly holds if $n = 1$. Suppose it holds for $n - 1$. Suppose $V = V_1 \cup \cdots \cup V_m$, with each V_i a proper subspace of V. Let W be a subspace of V of dimension $n - 1$. Then

$$W = (W \cap V_1) \cup \cdots \cup (W \cap V_m).$$

By the induction hypothesis, one of the subspaces on the right is not proper, i.e., $W \cap V_i = W$ for some i; this means $W \subset V_i$. By dimensions, $W = V_i$. We have proven that V contains only a finite number of subspaces of dimension $n-1$. This is false. For example, if $v_1, \ldots, v_n$ is a basis for V, then for each $t \in \mathbb{R}$ the spaces spanned by $v_1, \ldots, v_{n-2}, v_{n-1} + tv_n$ are distinct. □

Lemma 6.2. *Let G be a finite subgroup of* $\mathrm{O}(V)$ *and assume $G \neq 1$. Then there exists a point $x_0 \in V$ such that $Tx_0 \neq x_0$ for all $T \in G$, $T \neq 1$.*

Proof. For each $T \in G$, $T \neq 1$ consider the set V_T consisting of the fixed points of T acting on V, i.e., all the points $x \in V$ such that $Tx = x$. For $T \in G$, $T \neq 1$ the set V_T is a proper subspace of V. The previous lemma shows that the union of the V_T for $T \in G$, $T \neq 1$, cannot be all of V. Hence, there exists a point x_0 not in any of the V_T, $T \in G$, $T \neq 1$. □

Now assume $G \neq 1$ is a finite subgroup of $\mathrm{O}(V)$. We will describe a construction of a fundamental region based on a choice of a point x_0 as in the above lemma. Let

$$T_0 = 1, T_1, \ldots, T_{N-1}$$

be the elements of G. If we apply the elements of G to x_0, then we obtain

$$x_0 = T_0 x_0, \quad x_1 = T_1 x_0, \quad x_2 = T_2 x_0, \quad \ldots, \quad x_{N-1} = T_{N-1} x_0.$$

These points are distinct because if $T_i x_0 = T_j x_0$ for some $i, j \in \{0, 1, \ldots, N-1\}$, then $T_j^{-1} T_i x_0 = x_0$ with $T_j^{-1} T_i = T_k$ for some $k \in \{1, \ldots, N-1\}$; this contradicts the definition of x_0. These N points all lie on the sphere of radius $\|x_0\|$. Fix some i with $1 \le i \le N-1$. We consider the vector $x_0 - x_i$. As in Section 2, this vector defines a hyperplane

$$\mathcal{P}_i = (x_0 - x_i)^\perp$$

and two half-spaces:

$$\{x \in V : (x, x_0 - x_i) > 0\} \quad \text{and} \quad \{x \in V : (x, x_0 - x_i) < 0\}.$$

There are some other characterizations of $\mathcal{P}_i$ and these two half-spaces. Namely,

$$\mathcal{P}_i = \{x \in V : d(x, x_0) = d(x, x_i)\}.$$

To see this, we note that

$$\begin{aligned}
x \in \mathcal{P}_i &\Leftrightarrow (x, x_0 - x_i) = 0 \\
&\Leftrightarrow (x, x_0) = (x, x_i) \\
&\Leftrightarrow (x, x_0) = (x, T_i x_0) \\
&\Leftrightarrow -(x, x_0) - (x_0, x) = -(x, T_i x_0) - (T_i x_0, x) \\
&\Leftrightarrow (x, x) - (x, x_0) - (x_0, x) + (x_0, x_0) = (x, x) - (x, T_i x_0) - (T_i x_0, x) + (T_i x_0, T_i x_0) \\
&\Leftrightarrow (x - x_0, x - x_0) = (x - T_i x_0, x - T_i x_0) \\
&\Leftrightarrow d(x, x_0) = d(x, x_i).
\end{aligned}$$

Similarly,

$$\{x \in V : (x, x_0 - x_i) > 0\} = \{x \in V : d(x, x_0) < d(x, x_i)\}$$

and

$$\{x \in V : (x, x_0 - x_i) < 0\} = \{x \in V : d(x, x_0) > d(x, x_i)\}.$$

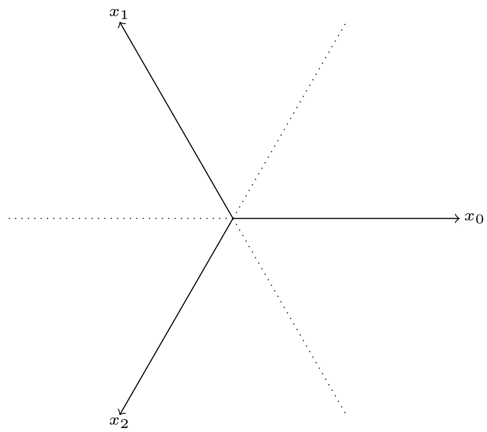
We will be particularly interested in the half-space

$$L_i = \{x \in V : d(x, x_0) < d(x, x_i)\}.$$

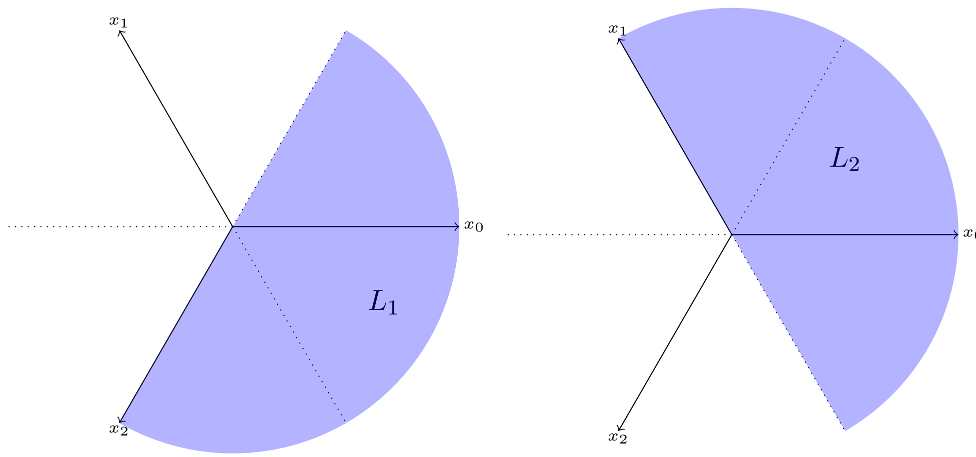
We set

$$F = \cap_{i=1}^{N-1} L_i.$$

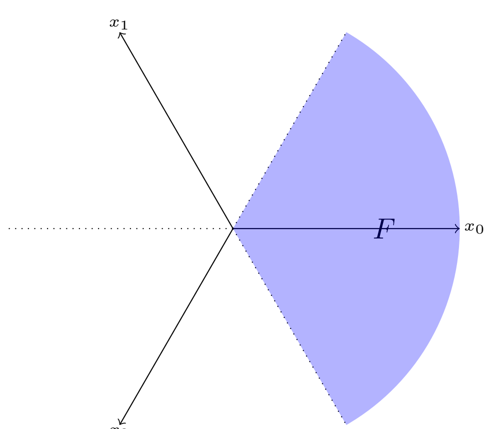
The set F is a convex cone extending to infinity. As an example, consider the case in the plane when $N = 3$, with say



Then we have:



so that



Theorem 6.3. *The set F is a fundamental region for G in V.*

Proof. We need to show that the three conditions are satisfied. First of all, it is clear that F is open because it is the intersection of a finite number of open sets.

Second, we need to show that for all $1 \leq i \leq N-1$ we have $F \cap T_i F = \varnothing$. To prove this, we compute $T_i F$. We have

$$\begin{aligned}
T_i F &= T_i \cap_{j=1}^{N-1} \{x \in V : d(x, x_0) < d(x, x_j)\} \\
&= T_i \{x \in V : d(x, x_0) < d(x, x_j), 1 \leq j \leq N-1\} \\
&= \{y \in V : \begin{array}{l} \text{there exists } x \in V \text{ such that } y = T_i x \text{ and} \\ d(x, x_0) < d(x, x_j) \text{ for } 1 \leq j \leq N-1 \end{array} \} \\
&= \{y \in V : \begin{array}{l} \text{there exists } x \in V \text{ such that } y = T_i x \text{ and} \\ d(T_i x, T_i x_0) < d(T_i x, T_i T_j x_0) \text{ for } 1 \leq j \leq N-1 \end{array} \} \\
&= \{y \in V : d(y, x_i) < d(y, T_k x_0), 0 \leq k \leq N-1, k \neq i\} \\
T_i F &= \{y \in V : d(y, x_i) < d(y, x_k), 0 \leq k \leq N-1, k \neq i\}.
\end{aligned}$$

If now $y \in F \cap T_i F$, then as $y \in F$ we have $d(y, x_0) < d(y, x_i)$, and as $y \in T_i F$, we have $d(y, x_i) < d(y, x_0)$: this is a contradiction.

Finally, we need to show that the union of the closures of the $T_i F$ is all of V. It is not hard to show that $1 \leq i \leq N-1$,

$$\overline{T_i F} = \{y \in V : d(y, x_i) \leq d(y, x_k), 0 \leq k \leq N-1\}.$$

Let $x \in V$. Choose i with $0 \leq i \leq N-1$ such that $d(x, x_i)$ is minimal. Then $d(x, x_i) \leq d(x, x_j)$ for all $0 \leq j \leq N-1$. By our characterization of $\overline{T_i F}$ we get $x \in \overline{T_i F}$. $\square$

7 Roots

We now let V be of arbitary finite dimension. Our overall goal is to classify the finite subgroups of $\mathrm{O}(V)$ generated by reflections, and we now introduce a geometric concept, called roots or the root system, which will be useful in this classification.

First we make a natural reduction. Let G be any subgroup of $\mathrm{O}(V)$. Let V_0 be the subset of all vectors x in V such that $Tx = x$ for all $T \in G$. The set V_0 is clearly a subspace of V, and we have a decomposition

$$V = V_0 \oplus V_0^{\perp}.$$

Since the elements of G preserve V_0 (they are the identity on V_0, the elements of G send $V_0^{\perp}$ into itself. We can thus regard every element T of G to be of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & T|_{V_0^{\perp}} \end{bmatrix},$$

and the map

$$G \hookrightarrow \mathrm{O}(V_0^{\perp}), \quad T \mapsto T|_{V_0^{\perp}}$$

is an injective homomorphism.

Proposition 7.1. *Let the notation be as above. If G is generated by reflections, then the image of G in $\mathrm{O}(V_0)$ is also generated by reflections.*

Proof. It will suffice to show that if $S \in G$ is a reflection through $\mathcal{P} = r^{\perp}$, then the image of S is also a reflection. Evidently, $V_0 \subset \mathcal{P}$. Hence, $\mathcal{P}^{\perp} \subset V_0^{\perp}$, that is, $r \in V_0^{\perp}$. Since $Sx = x - 2\frac{(x,r)}{(r,r)}r$ for $x \in V$, and since $r \in V_0^{\perp}$, $S|_{V_0^{\perp}}$ is also a reflection. $\square$

The last proposition shows that for the purposes of classifying subgroups of $\mathrm{O}(V)$ generated by reflections, it suffices to consider those subgroups such that $V_0 = 0$. We shall say that a subgroup $G \subset \mathrm{O}(V)$ is **effective** if $V_0 = 0$, i.e., if the elements of G do not have any common fixed points. A finite subgroup G of $\mathrm{O}(V)$ which is finite, effective and generated by reflections will be called a **Coxeter group**. Our goal is to classify Coexeter groups. Unless we say otherwise, in this section G will be a fixed Coxeter group.

In consider rotations in three space we found that poles were a very useful concept. An analogous concept exists for reflections. Namely, suppose $S \in G$ is a reflection through the hyperplane $\mathcal{P} = r^{\perp}$. We may assume that r has length one. We will call the vectors r and $-r$ the **roots** of S; the roots of G are the roots of the reflections in G.

Proposition 7.2. *If r is a root of G corresponding to the reflection S, and $T \in G$, then Tr is also a root of G. In fact, $S_{Tr} = TS_rT^{-1}$.*

Proof. It will suffice to prove $S_{Tr} = TS_rT^{-1}$. We have

$$\begin{aligned} TS_rT^{-1}x &= T(T^{-1} - 2\frac{(T^{-1}x, r)}{r, r)}r) \\ &= x - 2\frac{(T^{-1}x, r)}{(r, r)}Tr \\ &= x - 2\frac{(T^{-1}x, T^{-1}Tr)}{(Tr, Tr)}Tr \\ &= x - 2\frac{(x, Tr)}{(Tr, Tr)}Tr \\ &= S_{T_r}(x). \end{aligned}$$

This proves the proposition. □

Now fix a set of generators consisting of reflections for G, and let Δ be the set of consisting of the union of the orbits of these roots under the action of G. Since Δ is the union of orbits, G acts on Δ. We call Δ a **root system** for G. In fact, it will turn out later that Δ is the set of all the roots of G, so that there is only one root system associated to G, but for now we use this definition. Our intermediate goal is to prove some basic results about Δ.

Proposition 7.3. *The set Δ contains a spanning set, and hence a basis for V.*

Proof. Let
$$\Delta = \{x_1, \ldots, x_k\}.$$
Consider the subspace $W = \cap_{i=1}^k x_i^\perp$. As G is generated by reflections along the roots $x_1, \ldots, x_k$, and these reflections act trivially on the hyperplanes $x_i^\perp$, it follows that G acts trivially on W. Since G is effective, $W = 0$. This means

$$V = W^\perp = (\cap_{i=1}^k x_i^\perp)^\perp = x_1^{\perp\perp} + \cdots + x_k^{\perp\perp} = \mathbb{R}x_1 + \cdots + Rx_k.$$

(Here we have used $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$ which is equivalent to $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$; this follows because $x \in (W_1 + W_2)^\perp \iff (x, W_1 + W_2) = 0 \iff (x, W_1) = (x, W_2) = 0 \iff x \in W_1^\perp \cap W_2^\perp$.) Since $\{x_1, \ldots, x_k\}$ contains a spanning set, it contains a basis. □

Next, we geometrically partition Δ into positive and negative elements. This concept will depend on the choice of another auxiliary vector. Fix $t \in V$ such that $(t, r) \neq 0$ for all $r \in \Delta$. (Such a vector exists because V is not the union of the finitely many proper subspaces consisting of the kernels of $x \mapsto (x, r)$ for $r \in \Delta$). The root system Δ is partitioned into two sets:

$$\Delta^+ = \{r \in \Delta : (t, r) > 0\}, \quad \Delta^- = \{r \in \Delta : (t, r) < 0\}.$$

Geometrically what we are doing is taking the hyperplane $t^\perp$; the elements in Δ^+ lie on the same side of the hyperplane as t and form an acute angle with t; the elements in Δ^- lie on the side of the hyperplane not containing t and form an obtuse angle with t. Note that the positive roots Δ^+ and the negative roots Δ^- have the same number of elements; in fact, there is a bijection between these two sets given by $r \mapsto -r$.

We single out a subset of the positive roots that will contain a great deal of information about G. Choose a subset Π of the positive roots Δ^+ such that every element of Δ^+ can be written as a sum of elements from Π with nonnegative coefficients, and the number of elements of Π is minimal with respect to this property; such a subset Π is called a **base** for Δ. Such a Π exists, as Δ^+ is finite; after developing some ideas we will show that Π is unique. Let

$$\Pi = \{r_1, \ldots, r_m\}.$$

We will say that $x \in V$ is **positive** if x can be written as a linear combination of the elements from the base Π with nonnegative coefficients; we say that $x \in V$ is **negative** if it can be written as a linear combination of the elements from Π with nonpositive coefficients. Clearly, every element of Δ^+ is indeed positive and every element of Δ^- is negative. If x is positive then we have $(x, t) \geq 0$; if x is negative, we get $(x, t) \leq 0$.

Lemma 7.4. *Let $r_i, r_j \in \Pi$ with $i \neq j$. Let λ_i and λ_j be positive real numbers. Then $x = \lambda_i r_i - \lambda_j r_j$ is neither positive nor negative.*

Proof. Suppose x is positive. Then we can write

$$\lambda_i r_i - \lambda_j r_j = \mu_1 r_1 + \cdots + \mu_m r_m$$

with $\mu_1 \geq 0, \ldots, \mu_m \geq 0$. Suppose $\lambda_i \leq \mu_i$. Then

$$0 = (\mu_i - \lambda_i) r_i + (\mu_j + \lambda_j) r_j + \sum_{k=1,\ldots,m,\ k \neq i,j} \mu_k r_k$$

Since all the coefficients of this sum are nonnegative, if we take the inner product with t we get

$$0 = \Big(t, (\mu_i - \lambda_i) r_i + (\mu_j + \lambda_j) r_j + \sum_{k=1,\ldots,m,k \neq i,j} \mu_k r_k\Big) \geq \lambda_j (t, r_j) > 0.$$

This is a contradiction. Suppose $\lambda_i > \mu_i$. Then

$$(\lambda_i - \mu_i) r_i = (\mu_j + \lambda_j) r_j + \sum_{k=1,\ldots,m,k \neq i,j} \mu_k r_k.$$

This contradicts the minimality of Π. Thus, x is not positive. If x were negative, then $-x$ would be positive; a similar argument to the one above would give a contradiction. $\square$

Using this lemma, we can show that r_i and r_j form an obtuse angle:

Lemma 7.5. *Let $r_i, r_j \in \Pi$, with $i \neq j$. Then $(r_i, r_j) \leq 0$.*

Proof. To prove this we use the reflection $S_i \in G$ along r_i (S_i exists by definition). We know that $S_i r_j \in \Delta$ (by the definition of Δ). Hence, $S_i r_j$ is either positive or negative. Now

$$S_i r_j = r_j - 2\frac{(r_j, r_i)}{(r_i, r_i)} r_i.$$

One of the coefficients in the linear combination on the right hand side of this equation is positive, namely the coefficient of r_j which is 1. By the last lemma, the second coefficient must be nonnegative. This implies $(r_j, r_i) \leq 0$ (and $S_i r_j$ is positive). □

The importance of obtuseness is revealed in the next lemma:

Lemma 7.6. *Let $x_1, \ldots, x_m \in V$, and suppose these vectors all lie on the same side of a hyperplane, i.e., there exists an $x \in V$ such that $(x_i, x) > 0$ for $1 \leq i \leq m$. If x_i and x_j form an obtuse angle for $i \neq j$ then $\{x_1, \ldots, x_m\}$ is linearly independent.*

Proof. Suppose there is a nontrivial linear relation between the x_i. Then there exists an equation

$$\lambda_1 x_1 + \cdots + \lambda_k x_k = \mu_{k+1} x_{k+1} + \cdots + \mu_m x_m$$

with $\lambda_1, \ldots, \lambda_k \geq 0$, $\mu_{m+1}, \ldots, \mu_m \geq 0$ and say $\lambda_1 > 0$. Computing the square of the norm of this gives:

$$\begin{aligned} 0 \leq \|\lambda_1 x_1 + \cdots + \lambda_k x_k\|^2 &= (\lambda_1 x_1 + \cdots + \lambda_k x_k, \lambda_1 x_1 + \cdots + \lambda_k x_k) \\ &= (\lambda_1 x_1 + \cdots + \lambda_k x_k, \mu_{k+1} x_{k+1} + \cdots + \mu_m x_m) \\ &= \sum \lambda_i \mu_j (x_i, x_j) \leq 0. \end{aligned}$$

This implies

$$\lambda_1 x_1 + \cdots + \lambda_k x_k = 0.$$

This used the obtuse of the angles. Now we use that all the vectors lie on one side of a hyperplane. We have

$$0 = (\lambda_1 x_1 + \cdots + \lambda_k x_k, x) = \lambda_1 (x_1, x) + \cdots + \lambda_k (x_k, x) > 0.$$

This is a contradiction. □

Proposition 7.7. *The set Π is a basis for V. In particular, it contains n elements. There is only one base for Δ.*

Proof. The previous lemma shows that Π is linearly independent. We showed before Δ contains a spanning set; since every element of Δ can be written a linear combination of elements from Π, it follows that Π is a spanning set. Thus, Π is a basis for V.

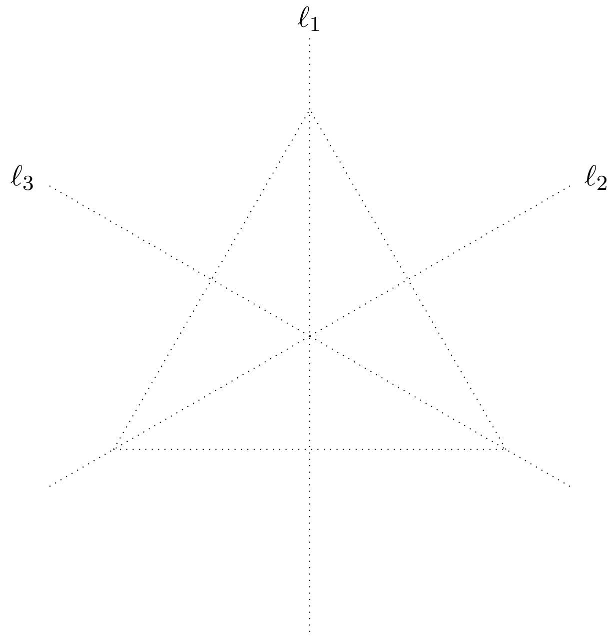
Next, suppose Π' is another base for Δ. Then Π' is also a basis for V. Let A be the change of basis matrix from Π to Π' and let B be the change of basis matrix from Π' to Π. We have $AB = 1$. Also, by the definition of a base, the entries of A and B are nonnegative real numbers. Let $a_1, \dots, a_n$ be the rows of A and $b_1, \dots, b_n$ the columns of B. Since $AB = 1$, we have

$$(a_1, b_2) = \dots (a_1, b_n) = 0.$$

It follows that a_1 has exactly one nonzero (which is nonnegative) entry (it must have at least one nonzero entry, as the rows of A are linearly independent. Suppose it has at least two nonzero entries. Then by these equations, the corresponding entries of $b_2, \dots, b_n$ are zero; recall that A and B have nonnegative entries. This implies that $b_2, \dots, b_n$ are not linearly independent, a contraction). Similar arguments show that each row has exactly one nonzero entry, which is positive. Recalling the meaning of A, it follows that each element of Π' is a positive multiple of an element of Π. Since roots have length one, it follows that this positive scalar is one. This means $\Pi' = \Pi$. $\square$

We call the elements of $\Pi = \{r_1, \dots, r_n\}$ the **fundamental roots** or **simple roots**. We call the reflections $S_1, \dots, S_n \in G$ along the roots $r_1, \dots, r_n$ the **fundamental reflections**. Our next goal is to prove that the fundamental reflections generate G.

First, however, we consider an example. Consider $G = \mathcal{H}_2^3 \subset \mathrm{O}(\mathbb{R}^2)$. This the group of symmetries of an equilateral triangle. We will orient our triangle as follows:



The group G is generated by R, the rotation by $2\pi/3$, and the reflection S through the vertical dotted line ℓ_1. Then the group is

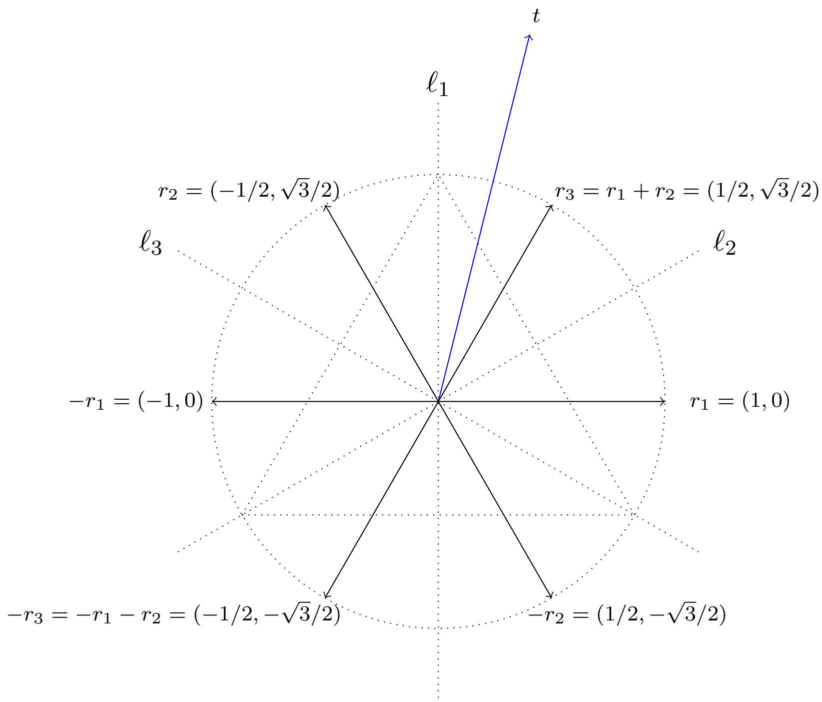
$$G = \{1, R, R^2, S, SR, SR^2\}.$$

The reflections in G are S, SR and SR^2. Here, SR is the reflection through the line ℓ_2 and SR^2 is the reflection through the line ℓ_3. The group G is generated by S and SR, and we take $\{S, SR\}$ as our generator set. Let r_1 and $-r_1$ be the roots of S, let r_2 and $-r_2$ be the roots of SR, and let r_3 and $-r_3$ be the roots of SR^2. We have $r_3 = r_1 + r_2$. Referring to the diagram below, we have:

$$\begin{aligned} 1 \cdot r_1 &= r_1 & 1 \cdot r_2 &= r_2, \\ Rr_1 &= r_2 & Rr_2 &= -r_3, \\ R^2 r_1 &= -r_3, & R^2 r_2 &= r_1, \\ Sr_1 &= -r_1, & Sr_2 &= r_3, \\ SRr_1 &= r_3, & SRr_2 &= -r_2, \\ SR^2 r_1 &= -r_2, & SR^2 r_2 &= -r_1. \end{aligned}$$

It follows that

$$\Delta = \{r_1, r_2, r_3 = r_1 + r_2, -r_1, -r_2, -r_3 = -r_1 - r_2\}.$$



Let t be as in the above diagram. Then we get:

$$\Delta^+ = \Delta_t^+ = \{r_1, r_2, r_1 + r_2\}, \quad \Delta^- = \Delta_t^- = \{-r_1, -r_2, -(r_1 + r_2)\}$$

and

$$\Pi = \{r_1, r_2\}.$$

Proposition 7.8. *Let $r_i \in \Pi = \{r_1, \ldots, r_n\}$, and let $S_i \in G$ be the reflection along r_i. If $r \in \Delta^+$ and $r \neq r_i$, then $S_i r \in \Delta^+$.*

Proof. Since r is a root, we know that $S_i r$ is another root. That is, $S_i r \in \Delta$. Thus, $S_i r$ is positive or negative. Write

$$r = \lambda_1 r_1 + \cdots + \lambda_n r_n.$$

Since r is positive, $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ with at least one of these nonzero. We have

$$\begin{aligned} S_i r &= r - 2(r, r_i) r_i \\ &= (\lambda_1 r_1 + \cdots + \lambda_n r_n) - 2(\lambda_1 r_1 + \cdots + \lambda_n r_n, r_i) r_i \\ &= \big(\lambda_i - 2(\lambda_1 r_1 + \cdots + \lambda_n r_n, r_i)\big) r_i + \sum_{j=1, j\neq i}^{n} \lambda_j r_j \end{aligned}$$

If some λ_j with $j \neq i$ is nonzero, then $S_i r$ is positive. Assume $\lambda_j = 0$ for $j \neq i$. Then

$$\begin{aligned} S_i r &= \big(\lambda_i - 2(\lambda_1 r_1 + \cdots + \lambda_n r_n, r_i)\big) r_i + \sum_{j=1, j\neq i}^{n} \lambda_j r_j \\ &= \big(\lambda_i - 2(\lambda_i r_i, r_i)\big) r_i \\ &= -\lambda_i r_i \\ &= -r. \end{aligned}$$

Since $S_i r = -r$, r is a multiple of r_i. Since r and r_i are positive and have length 1, it follows that $r = r_i$. But we assumed $r \neq r_i$; this contradiction completes the proof. $\square$

Proposition 7.9. *Let $r \in \Delta^+$. Then there exists a product T of fundamental reflections such that $Tr \in \Pi$.*

Proof. If $r \in \Pi$ we take $T = 1$. Assume $r \notin \Pi$. Consider the set $\Pi \cup \{r\}$. This set is not linearly independent. On the other hand, all of these vectors lie on one side of the hyperplane $t^\perp$. By Lemma 7.5 and Lemma 7.6, there exists an i such that $(r, r_i) > 0$: one of simple roots makes an acute angle with r. Let's apply the fundamental reflection S_i to r. By the Proposition 7.8 (note that $r \notin \Pi$), we know that

$$S_i r \in \Delta^+.$$

And this vector also makes a less acute angle with t than does r:

$$(S_i r, t) = (r - 2(r, r_i) r_i, t) = (r, t) - 2(r, r_i)(r_i, t) < (r, t).$$

If $S_i r$ is in Π we can take $T = S_i$. Suppose $S_i r \notin \Pi$. Then we can repeat what we just did and find a j such that $S_j S_i r \in \Delta^+$ and

$$(S_j S_i r, t) < (S_i, t) < (r, t).$$

But we cannot continue this process forever: if we could, then $r, S_i r, S_j S_i r, \ldots$ would be distinct elements of Δ^+, and Δ^+ is finite (here we use the finite generation of G). Thus, the process terminates at some point, which proves the proposition. $\square$

Theorem 7.10. *The fundamental reflections $S_1, \ldots, S_n$ generate G.*

Proof. By definition, G is generated by the S_r for $r \in \Delta$. Also, $S_r = S_{-r}$ for $r \in \Delta$. Hence, it suffices to prove that the S_r for $r \in \Delta^+$ can be written as products of fundamental reflections. Let $r \in \Delta^+$. By the last proposition, there exists a product T of fundamental reflections such that $Tr \in \Pi$. Write $Tr = r_i$. We have

$$S_r = S_{Tr_i} = T S_{r_i} T^{-1}.$$

This proves the theorem. $\square$

Thus, we have proven that for every Coxeter group there exists a special basis Π for V consisting of roots which form obtuse angles with each other, and the reflections associated with roots generate the group. Our next goal is to determine a very natural fundamental domain for G which also involves Π. A consequence of this will be that the root system, which was initially defined to be smaller than the set of all roots, is in fact equal to the set of all roots.

Proposition 7.11. *If $T \in G$ and $T\Pi = \Pi$, then $T = 1$.*

Proof. Suppose $T \neq 1$; we will obtain a contradiction. Write

$$T = S_{i_1} S_{i_2} \dots S_{i_k};$$

we may assume that T cannot be written as a product of a smaller number of fundamental reflections. Now since $T\Pi = \Pi$ we have

$$Tr_{i_k} = S_{i_1} \dots S_{i_k} r_{i_k} = -S_{i_1} \dots S_{i_{k-1}} r_{i_k} \in \Pi.$$

This implies that

$$S_{i_1} \dots S_{i_{k-1}} r_{i_k} \in \Delta^-.$$

Consider the sequence of roots:

$$\begin{aligned} a_0 &= S_{i_1} \dots S_{i_{k-1}} r_{i_k}, \\ a_1 &= S_{i_2} \dots S_{i_{k-1}} r_{i_k}, \\ a_2 &= S_{i_3} \dots S_{i_{k-1}} r_{i_k}, \\ &\vdots \\ a_{k-1} &= r_{i_k}. \end{aligned}$$

The first root, a_0, is in Δ^-, while the last root, a_{k-1} is in Δ^+. Let a_j be the first of the roots to be positive. We have

$$a_j = S_{i_{j+1}} \dots S_{i_{k-1}} r_{i_k} = S_{i_j} S_{i_j} S_{i_{j+1}} \dots S_{i_{k-1}} r_{i_k} = S_{i_j} a_{j-1}.$$

and therefore

$$a_{j-1} = S_{i_j} a_j.$$

The root a_j is positive, while a_{j-1} is negative. By Proposition 7.8 we must have

$$r_{i_j} = a_j.$$

We now have the equation

$$r_{i_j} = S_{i_{j+1}} \dots S_{i_{k-1}} r_{i_k}.$$

By Proposition 7.2 this gives

$$S_{i_j} = S_{r_{i_j}} = S_{i_{j+1}} \dots S_{i_{k-1}} S_{i_k} (S_{i_{j+1}} \dots S_{i_{k-1}})^{-1}.$$

This is equivalent to

$$S_{i_j} S_{i_{j+1}} \dots S_{i_{k-1}} = S_{i_{j+1}} \dots S_{i_{k-1}} S_{i_k}.$$

This gives

$$\begin{aligned} T &= S_{i_1} \dots S_{i_k} \\ &= S_{i_1} \dots S_{i_{j-1}} S_{i_j} S_{i_{j+1}} \dots S_{i_{k-1}} S_{i_k} \\ &= S_{i_1} \dots S_{_{j-1}} S_{i_{j+1}} \dots S_{i_{k-1}} S_{k_i} S_{k_i} \\ &= S_{i_1} \dots S_{_{j-1}} S_{i_{j+1}} \dots S_{i_{k-1}}. \end{aligned}$$

This contradicts the minimality of k. □

So far, we have suppressed the dependence of Δ^+ and Δ^- on t, but now recall it and use it as a tool.

Proposition 7.12. *If $T \in G$, then $T(\Delta_t^+) = \Delta_{T(t)}^+$ and $T(\Pi_t) = \Pi_{T(t)}$.*

Proof. We have

$$\begin{aligned} T(\Delta_t^+) &= T\{r \in \Delta : (t, r) > 0\} \\ &= \{x \in \Delta : x = Tr, r \in \Delta, (t, r) > 0\} \\ &= \{x \in \Delta : (t, T^{-1}x) > 0\} \\ &= \{x \in \Delta : (Tt, x) > 0\} \\ &= \Delta_{T(t)}^+. \end{aligned}$$

To see that $T(\Pi_t) = \Pi_{T(t)}$ we note first that $T(\Pi_t) \subset T(\Delta_t^+) = \Delta_{T(t)}^+$. Also, it is evident that every element of $T(\Delta_t^+) = \Delta_{T(t)}^+$ can be written as a linear combination of elements from $T(\Pi_t)$ and that $T(\Pi_t)$ has n elements. By Proposition 7.7 it follows that $T(\Pi_t)$ is a base for the choice $T(t)$; by the uniqueness of base, $T(\Pi_t) = \Pi_{T(t)}$. □

Proposition 7.13. *If $T \in G$ and $T(\Delta^+) = \Delta^+$, then $T = 1$.*

Proof. To prove this we will show that $T\Pi = \Pi$ and apply Proposition 7.11. We have

$$\Delta^+ = T\Delta^+ = T\Delta_t^+ = \Delta_{T(t)}^+.$$

By the uniqueness of base, $\Pi = \Pi_{T(t)}$. On the other hand, we just proved $T\Pi = \Pi_{T(t)}$. Putting these together, we get $T\Pi = \Pi$. □

Next, we describe the mentioned fundamental region for G. This is simply described as all the vectors in V which make a strictly acute angle with every fundamental root. That is, we will show that

$$F = \{x \in V : (x, r_1) > 0, \ldots, (x, r_n) > 0\}$$

is a fundamental region for G. Before proving that this is a fundamental region we will make a few observations about this set. First, F is the intersection of the open half-spaces determined by the fundamental roots:

$$F = \cap_{i=1}^{n} \{x \in V : (x, r_i) > 0\}.$$

Let

$$P_i = r_i^{\perp}$$

for $i = 1, \ldots, n$; these are the hyperplanes through the which the fundamental reflections reflect. The closure $\bar{F}$ is:

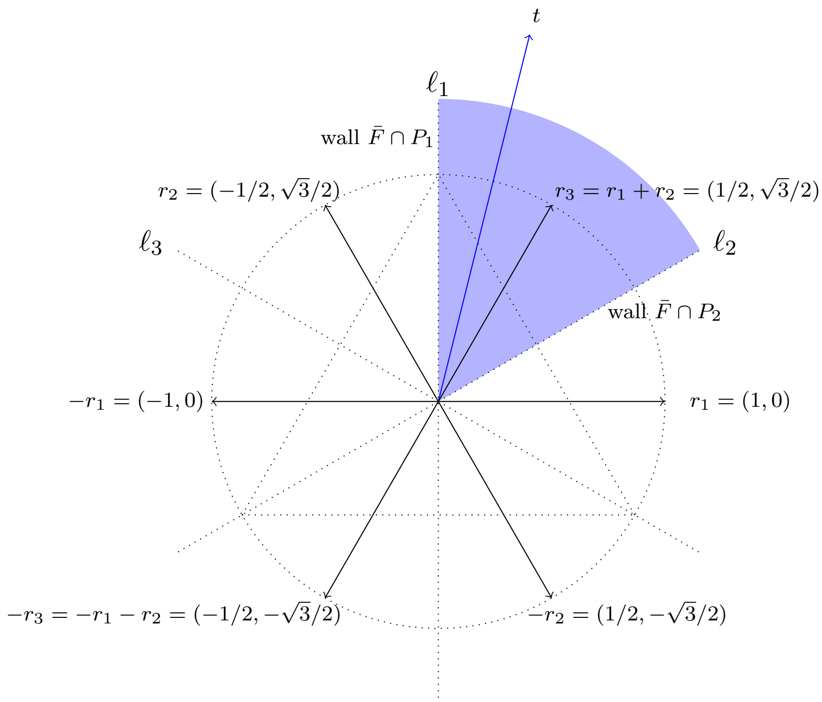
$$\bar{F} = \{x \in V : (x, r_1) \geq 0, \ldots, (x, r_n) \geq 0\} = \cap_{i=1}^{n} \{x \in V : (x, r_i) \geq 0\}.$$

The boundary of F is:

$$\text{boundary of } F \quad = (\bar{F} \cap P_1) \cup \cdots \cup (\bar{F} \cap P_n).$$

The sets $(\bar{F} \cap P_i)$ are called the **walls** of F. The fundamental reflection S_i is the reflection through the i-th wall of F.

Going back to our example,



Theorem 7.14. *The set F is a fundamental region for G.*

Proof. We need to prove three statements: F is open; $F \cap TF = \varnothing$ for $T \in G$ and $T \neq 1$; and $V = \cup_{T \in G} T\bar{F} = \cup_{T \in G} \overline{TF}$.

First, it is clear that F is open.

Next, suppose $x \in F \cap TF$; we need to prove $T = 1$. Since $x \in F$, we certainly have $(x, r_1) > 0, \ldots, (x, r_n) > 0$. This implies that $(x, r) > 0$ for all $r \in \Delta^+$. Hence, $\Delta^+ = \Delta_t^+ \subset \Delta_x^+$; by cardinalities, we have $\Delta_t^+ = \Delta_x^+$ (these set have the same cardinalities because their orders are half of the cardinality of Δ). By Proposition 7.7, $\Pi_t = \Pi_x$. Since $x \in TF$, we also have $T^{-1}x \in F$. This also gives $\Pi_{T^{-1}x} = \Pi_t$. Therefore,

$$\Pi_t = \Pi_{T^{-1}x} = T^{-1}\Pi_x = T^{-1}\Pi_t.$$

By Proposition 7.13, $T = 1$.

Finally, let $y \in V$. We need to prove that there exists $T \in G$ such that $(Tx, r_1) \geq 0, \ldots, (Tx, r_n) \geq 0$. To do this, let

$$x_0 = \frac{1}{2} \sum_{r \in \Delta^+} r.$$

Then for $1 \leq i \leq n$ we have, by Proposition 7.8,

$$S_i x_0 \quad = \quad S_i(\frac{1}{2} r_i + \frac{1}{2} \sum_{r \in \Delta^+, r \neq r_i} r)$$

$$= -\frac{1}{2}r_i + \frac{1}{2}\sum_{r\in\Delta^+, r\neq r_i} r$$
$$= x_0 - r_i.$$

What has this to do with our task? Let $T \in G$ be such that (Ty, x_0) is maximal. By this maximality and the above equation,

$$(Ty, x_0) \geq (S_iTy, x_0) = (Ty, S_ix_0) = (Ty, x_0 - r_i) = (Ty, x_0) - (Ty, r_i).$$

That is,

$$(Ty, r_i) \geq 0,$$

as desired. □

Theorem 7.15. *Every reflection in G is conjugate to a fundamental reflection. Every root is in the root system.*

Proof. We will show that if r is a root for G, then $r = \pm Tr_i$ for some $T \in G$ and i with $1 \leq i \leq n$. This will prove the theorem because then $S_r = S_{Tr_i} = TS_iT^{-1}$ by Proposition 7.2, proving that every reflection in G is conjugate to a fundamental reflection; and $r \in \Delta$ by the definition of Δ.

To prove our statement, let $P = r^\perp$. We claim first that $P \cap TF = \varnothing$ for all $T \in G$. To see this, suppose $P \cap TF \neq \varnothing$ for some $T \in G$. Let $x \in P \cap TF$. Then, of course, $x \in TF$. Also, we have $x = S_rx \in S_rTF$. This means that $TF \cap S_rTF \neq \varnothing$. But this contradicts the fact that TF is a fundamental region for G (note that $S_r \neq 1$).

So $P \cap \bigcup_{T\in G} TF = \varnothing$. In addition, there is a decomposition:

$$\bar{F} = F \cup (\bar{F} \cap P_1) \cup \cdots \cup (\bar{F} \cap P_n).$$

This implies that

$$V = \bigcup_{T\in G} TF \cup \bigcup_{T\in G} T(\bar{F} \cap P_1) \cup \cdots \cup \bigcup_{T\in G} T(\bar{F} \cap P_n).$$

This is the same as

$$V = \bigcup_{T\in G} TF \cup (\bigcup_{T\in G} T\bar{F} \cap TP_1) \cup \cdots \cup (\bigcup_{T\in G} T\bar{F} \cap TP_n).$$

Since $P \cap \cup_{T\in G} TF = \varnothing$ we conclude that

$$P \subset \bigcup_{T\in G, 1\leq i\leq n} TP_i.$$

This implies

$$P = \bigcup_{T\in G, 1\leq i\leq n} P \cap TP_i.$$

Now consider the $P \cap TP_i$. These are subspaces of P. One of these subspaces is not proper, by Lemma 6.1. Hence, for some T and i, $P = P \cap TP_i$. This means that $P \subset TP_i$. By dimensions, $P = TP_i$. Taking orthogonal complements, we get $r = \pm Tr_i$. □

There is another characterization of the fundamental region which we want to discuss.

Proposition 7.16. *Let $\Pi^* = \{s_1, \ldots, s_n\}$ be the basis for V dual to $\Pi = \{r_1, \ldots, r_n\}$, i.e., the basis such that $(s_i, r_j) = \delta_{ij}$ for $1 \le i, j \le n$. Then*

$$F = \{x \in V : x = \lambda_1 s_1 + \cdots + \lambda_n s_n, \lambda_1 > 0, \ldots \lambda_n > 0\}.$$

Proof. To see this, let $x \in V$ and write $x = \lambda_1 s_1 + \cdots + \lambda_n s_n$. Then

$$(x, r_i) = \lambda_i (s_i, r_i) = \lambda_i$$

for $1 \le i \le n$. □

There is some terminology associated with this point of view. The **convex hull** of $s_1, \ldots, s_n$ is the smallest convex subset of V containing these vectors. It is

$$\mathrm{co}(\Pi^*) = \{x \in V : x = \lambda_1 s_1 + \cdots + \lambda_n s_n, \lambda_1 \ge 0, \ldots, \lambda_n \ge 0, \lambda_1 + \cdots + \lambda_n = 1\}.$$

We have

$$\bar{F} = \mathbb{R}_{\ge 0} \cdot \mathrm{co}(\Pi^*).$$

The set $\mathrm{co}(\Pi^*)$ is also called the **simplex** spanned by $s_1, \ldots, s_n$, and $\mathbb{R}_{\ge 0} \cdot \mathrm{co}(\Pi^*)$ is called the **simplicial cone** spanned by $s_1, \ldots, s_n$.

We can say a bit more about $s_1, \ldots, s_n$: they form acute angles.

Theorem 7.17. *Let $\{r_1, \ldots, r_n\}$ be a basis for V and assume $(r_i, r_j) \le 0$ for $i \ne j$. Let $\{s_1, \ldots, s_n\}$ be the dual basis. Then $(s_i, s_j) \ge 0$ for all i, j.*

Proof. Let A be the change of basis matrix from the basis $\{s_1, \ldots, s_n\}$ to $\{r_1, \ldots, r_n\}$, and let B be the change of basis matrix from the basis $\{r_1, \ldots, r_n\}$ to $\{s_1, \ldots, s_n\}$. Of course, we have

$$AB = 1.$$

What are the entries of A and B? Let $A = (A_{ij})$. By definition,

$$r_j = \sum_{l=1}^{n} A_{lj} s_l.$$

Then

$$(r_i, r_j) = \sum_{l=1}^{n} A_{lj} (r_i, s_l) = A_{ij}.$$

We thus find that

$$A = (r_i, r_j)_{1 \le i,j \le n}, \quad B = A^{-1} = (s_i, s_j)_{1 \le i,j \le n}.$$

We need to show that the entries of B are nonnegative. To do this, we need to gather some information about A. First of all, it is clear that A is symmetric. It is a fact from linear algebra that every symmetric real matrix can be diagonalized. So we can find a matrix U such that

$$UAU^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Let λ be an eigenvalue of A with eigenvector x. Write

$$x = d_1 r_1 + \cdots + d_n r_n.$$

Since the matrix A is also the matrix of the positive definite symmetric bilinear form $(\cdot, \cdot)$ on V in the ordered basis $r_1, \ldots, r_n$ we have

$$\begin{aligned} (x, x) &= \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix} A \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \\ &= \lambda \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \\ (x, x) &= \lambda \sum_{i=1}^{n} d_i^2. \end{aligned}$$

As $(x, x) > 0$ and $\sum_{i=1}^n d_i^2$, it follows that $\lambda > 0$. Also, because $(r_1, r_1) = \cdots = (r_n, r_n) = 1$, the trace of A is n and

$$\lambda_1 + \cdots + \lambda_n = 1.$$

From this we conclude that $0 < \lambda_1, \ldots, \lambda_n < 1$. Now introduce the matrix

$$C = I - (1/n)A.$$

The entries of C are nonnegative because the entries of A are nonpositive. We have

$$UCU^{-1} = 1 - (1/n)UAU^{-1} = \begin{bmatrix} 1 - \lambda_1/n & & \\ & \ddots & \\ & & 1 - \lambda_n/n \end{bmatrix}.$$

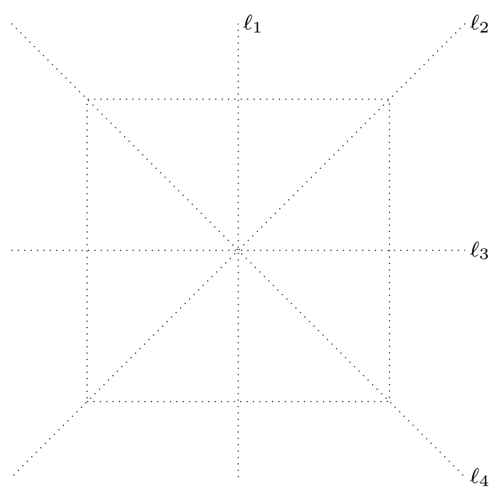
Thus, C is also diagonalizable with eigenvalues that are between 0 and 1. We get

$$\begin{aligned} A &= n(1-C) \\ A^{-1} &= (1/n)(1-C)^{-1} \\ B &= (1/n)(I + C + C^2 + \dots). \end{aligned}$$

Here, the infinite sum $1 + C + C^2 + \cdots$ does converge because C has eigenvalues between 0 and 1. Since all the entries of C are nonnegative, by the above equality all the entries of B are nonnegative. □

Before consider another example we make a comment about what we have proven and other possible choices for the lengths of roots. We started with a Coxeter group G. If Δ is the set of all the roots of G, then we introduced the concept of positive roots Δ^+ (which depends on a choice t), we proved that Δ^+ has a base Π, that the cardinality of Π is n, that G is generated by the reflections corresponding to the fundamental roots, and there is a natural fundamental region. Suppose that instead of initially choosing the set of roots to all have length one we chose other lengths for the roots. Then the definition of Δ would change, the set of positive roots would be multiplies of the old positive roots, the set of fundamental roots would be multiples of the old fundamental roots, the set of fundamental reflections would be the same, and so would the fundamental region. So changing the lengths of the roots would have no significant effect on the results we proved; and it will be convenient to make other choices for lengths, as in the following example.

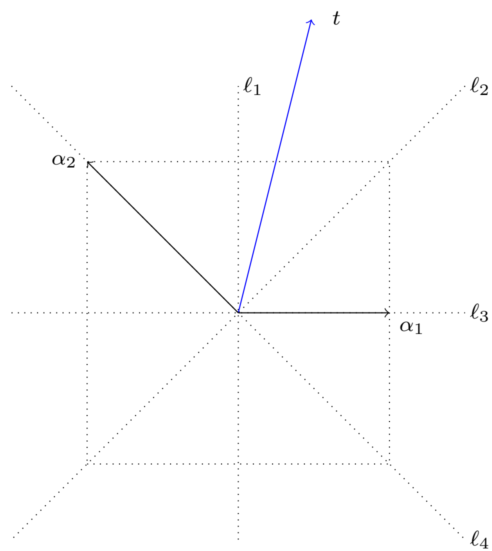
We consider $H_2^4 \subset \mathrm{O}(\mathbb{R}^2)$. These are the symmetries of the square:



Let R be rotation in the counterclockwise direction by $2\pi/4 = \pi/2$ degrees, and let S be the reflection through the line ℓ_1. The group is

$$H_2^4 = \{1, R, R^2, R^3, S, SR, SR^2, SR^3\}.$$

The reflections in this group are S, SR, SR^2 and SR^3. The reflection SR is the reflection through the line ℓ_2, the reflection SR^2 is the reflection through the line ℓ_3, and SR^3 is the refection through the line ℓ_4. The group G is generated by S and SR. Let α_1 and α_2 be the following roots for S and SR, and let t be as in the following diagram:



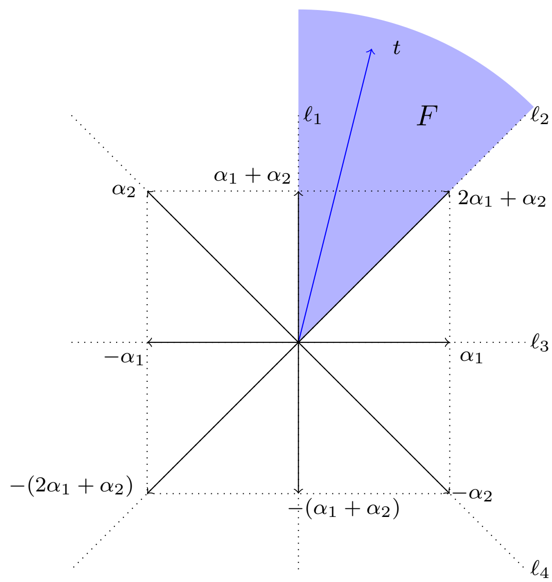
The roots are then

$$\begin{aligned} S: &\quad \pm\,\alpha_1, \\ SR: &\quad \pm\,\alpha_2, \\ SR^2: &\quad \pm\,(\alpha_1+\alpha_2), \\ SR^3: &\quad \pm\,(2\alpha_1+\alpha_2), \end{aligned}$$

so that

$$\begin{aligned} \Delta &= \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, -\alpha_1, -\alpha_2, -\alpha_1-\alpha_2, -2\alpha_1-\alpha_2\}, \\ \Delta^+ &= \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}, \\ \Delta &= \{-\alpha_1, -\alpha_2, -\alpha_1-\alpha_2, -2\alpha_1-\alpha_2\}, \\ \Pi &= \{\alpha_1, \alpha_2\}, \end{aligned}$$

as in the following diagram:



t
ℓ_1
F
ℓ_2
$\alpha_1 + \alpha_2$
α_2
$2\alpha_1 + \alpha_2$
ℓ_3
$-\alpha_1$
α_1
$-(2\alpha_1 + \alpha_2)$
$-\alpha_2$
$-(\alpha_1 + \alpha_2)$
ℓ_4

8 Classification of Coxeter groups

In this section G is Coxeter group with root system Δ and $\Pi = \{r_1, \ldots, r_n\}$ is a base. We recall that a Coxeter is a finite subgroup of $\mathrm{O}(V)$ which is generated by reflections, and which has the further property of being effective, i.e., there is no nonzero vector fixed by all the elements of G. If a group is not effective then it can be embedded into $\mathrm{O}(V')$ for V' of dimension $n-1$.

We begin our classification of Coxeter groups by making a further reduction. We say that a Coxeter group is **irreducible** if Π cannot be written as the union $\Pi_1 \cup \Pi_2$ of two nonempty subsets Π_1 and Π_2 with all the vectors in Π_1 orthogonal to all the vectors in Π_2.

Lemma 8.1. *If $\Pi = \Pi_1 \cup \Pi_2$ with $\Pi_1 \neq \varnothing$, $\Pi_2 \neq \varnothing$ and $\Pi_1 \perp \Pi_2$, then $G = G_1 \times G_2$, for some Coxeter groups $G_1 \neq 1$ and $G_2 \neq 1$.*

Proof. Let V_1 be the subspace of V spanned by Π_1 and let V_2 be the subspace of V spanned by Π_2. Then we have an orthogonal decomposition:

$$V = V_1 \perp V_2.$$

Suppose that S_i is a reflection corresponding to a simple root $r_i \in \Pi_1$. Let us consider the action of S_i on V_1 and V_2. To understand the action of S_i on V_2 is suffices to understand the action of S_i on the elements in Π_2. Let $r_j \in \Pi_2$. By assumption, $(r_i, r_j) = 0$. Using the formula for a reflection, we have

$$S_i r_j = r_j - 2\frac{(r_j, r_i)}{(r_i, r_i)} r_i = r_j.$$

It follows that S_i is the identity on V_2. In particular, S_i maps V_2 to itself. This implies that S_i maps $V_1 = V_2^{\perp}$ to itself. Similarly, if S_j is a reflection corresponding to a simple root $r_j \in \Pi_2$, then S_j is the identity on V_1 and maps V_2 to itself. Therefore, as the elements of G are generated by the simple reflections, every element of G maps V_1 to itself and V_2 to itself; its matrix has the form:

$$\begin{bmatrix} g|_{V_1} & 0 \\ 0 & g|_{V_2} \end{bmatrix}.$$

We define a map

$$i : G \to \mathrm{O}(V_1) \times \mathrm{O}(V_2)$$

by

$$i(g) = (g|_{V_1}, g|_{V_2}).$$

This map is clearly an injective homomorphism. Also, let

$$G_1 = \{g|_{V_1} : g \in G\}, \quad G_2 = \{g|_{V_2} : g \in G\}.$$

We claim that

$$i(G) = G_1 \times G_2.$$

It is clear that $i(G) \subset G_1 \times G_2$. To see that $G_1 \times G_2 \subset G$, it will suffice to show $G_1 \times 1 \subset i(G)$ and $1 \times G_2 \subset i(G)$. Consider an element g_1 of G_1. It is obtained by restricting an element of G to V_1. As the simple reflections generate G, g is of the form

$$g_1 = S_1|_{V_1} \cdots S_t|_{V_1}$$

where $S_1, \ldots, S_t$ are simple reflections corresponding to the elements of Π. Since all the simple reflections corresponding to the elements of V_2 restrict to the identity on V_1, we may assume that $S_1, \ldots, S_t$ correspond to elements of Π_1. Since the simple reflections corresponding to elements of Π_1 restrict to the identity on V_2, we get

$$i(S_1 \cdots S_t) = (g_1, 1).$$

It follows that $G_1 \times 1 \subset i(G)$. Similarly, $1 \times G_2 \subset i(G)$, and we get $G_1 \times G_2 = i(G)$.

Finally, we need to see that G_1 and G_2 are Coxeter groups. First of all, if S_i is the simple reflection corresponding to an element $r_i \in \Pi_1$, then restriction of S_i to V_1 is also a reflection: this follows from the formula for S_i, which also holds on V_1 – note that $r_i \in V_1$. Hence, G_1 is generated by reflections; similarly, G_2 is generated by reflections. If all the elements of G_1 fixed a vector v_1 in V_1, then all the elements of G would fix the vector $v_1 \oplus 0$ in $V = V_1 \oplus V_2$; hence, $v_1 \oplus 0 = 0$, and $v_1 = 0$. Therefore, G_1 is effective; similarly, G_2 is effective. This completes the proof. □

Because of the lemma we will often assume that G is irreducible.

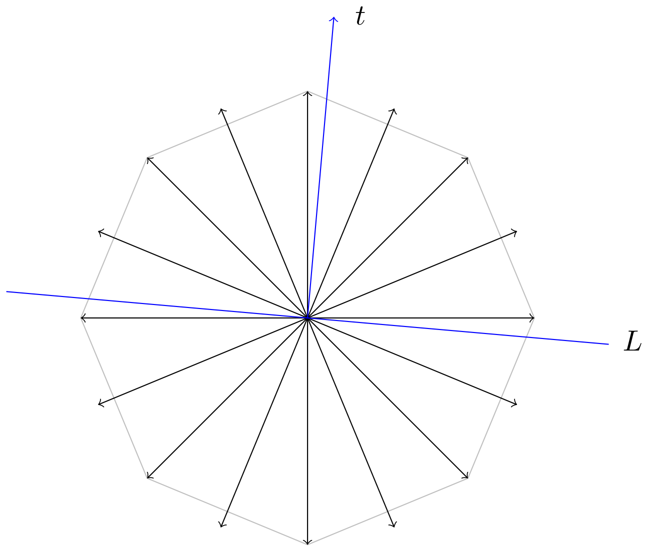
Next, we move on the classification of irreducible Coxeter groups. The following lemma will be useful in proving a fundamental result about the angles made by simple roots.

Lemma 8.2. *Let H be a dihedral group in $\mathrm{O}(\mathbb{R}^2)$ of order $2m$. Suppose that t has been chosen, and $\{r_1, r_2\}$ is the corresponding base, and $\{s_1, s_2\}$ is the basis dual to $\{r_1, r_2\}$. Then the angle φ between s_1 and s_2 is π/m, and if θ is the angle between r_1 and r_2, then*

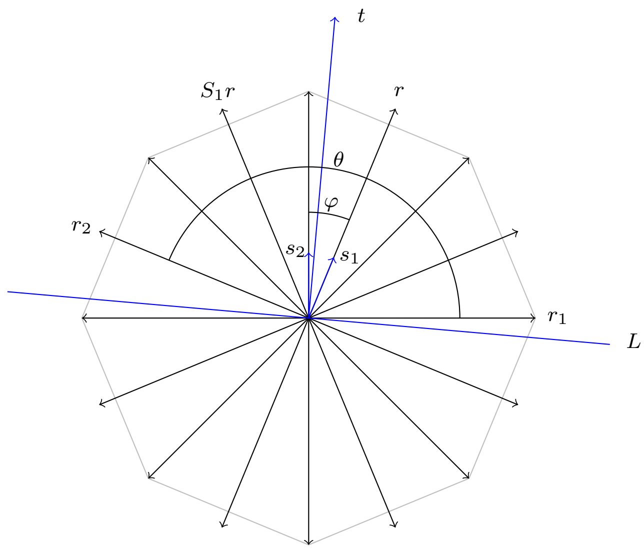
$$\theta + \varphi = \pi.$$

The order of $S_1 S_2$ is m.

Proof. The group H is the symmetry group of a regular m-gon centered at the origin. We will assume that m is even; the proof when m is odd is similar. The roots of H pass through the vertices and midpoints of the sides of the m-gon. By definition, t is not orthogonal to any root. Hence, the line L orthogonal to t does not contain a root. As an example, consider the case $m = 8$:



The positive roots are the roots in the half-plane determined by L which contains t, i.e., all the roots which make an acute angle with t. The base $\{r_1, r_2\}$ consists of the two positive roots nearest L, and the vectors s_1 and s_2 are scalar multiples of the roots as indicated in the following picture when $m = 8$:



It is evident that $\varphi = 2\pi/2m = \pi/m$ and $\theta + \varphi = \pi$. The element S_1S_2 is a rotation and sends the positive root r on the line generated by s_1 to the positive root S_1r; hence, S_1S_2 is a rotation through an angle $2\pi/m$. Therefore, S_1S_2 has order m. □

Theorem 8.3. *Let G be a Coxeter group. Let $r_i, r_j \in \Pi$. Then there exists an integer $p_{ij} \geq 1$ such that*

$$\frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = (r_i, r_j) = -\cos(\pi/p_{ij}) = \cos(\frac{p_{ij}-1}{p_{ij}}\pi).$$

The integer p_{ij} is the order of S_iS_j.

Proof. If $i = j$ then $(r_i, r_j)/\|r_i\|\|r_j\| = 1$, and we take $p_{ij} = 1$. Assume $i \neq j$. Consider the subspace W spanned by r_i and r_j; this is two dimensional. We have a decomposition of vector spaces:

$$V = W \oplus W^{\perp}.$$

Let H be the subgroup of G generated by S_i and S_j. Now $r_i \in W$ and so $W^{\perp} \subset r_i^{\perp}$. Since S_i is the identity on $r_i^{\perp}$, it follows that $S_i|_{W^{\perp}} = 1$; similarly, $S_j|_{W^{\perp}} = 1$. In particular, S_i and S_j map $W^{\perp}$ to $W^{\perp}$, and hence map W to W. It follows that the elements of H map W to W. It follows that the map

$$i : H \to \mathrm{O}(W), \quad S \mapsto S|_W$$

is well-defined. This homomorphism is also injective: if an element maps to the identity, then it is trivial as it is trivial on $W^{\perp}$. The elements S_i and S_j map to reflections in $\mathrm{O}(W)$ as they have determinant -1. Hence, $i(H)$ in $\mathrm{O}(W)$ is generated by the reflections S_i and S_j; by Theorem 4.4, $i(H)$ is a dihedral group of order, say, $2m$. The vectors $r_i, r_j \in W$ are roots for $i(H)$; we claim that there exists a t in W such that $\{r_i, r_j\}$ is a t-base for $i(H)$. Suppose not. Since r_i and r_j form a strictly obtuse angle θ ($\pi/2 \leq \theta < \pi$; the last inequality follows because otherwise $(r_i, r_j) = -1$, implying that $(r_i + r_j, r_i + r_j) = 0$ so that $r_i = -r_j$, contradicting the assumption that both are positive), and by our knowledge of dihedral groups and their root systems, there exists a t' and another root r for $i(H)$ such that r, r_i, r_j are positive roots and $\{r, r_j\}$ is a t'-base for $i(H)$. We claim that r is a root for G. To see this we first note that since r is a root for $i(H)$, there exists $S \in i(H) \subset \mathrm{O}(W)$ such that S is the reflection with respect to $r^{\perp}$, where $r^{\perp}$ is taken inside W. Let S' be the reflection on V with respect to the hyperplane $r^{\perp} \oplus W^{\perp}$. We have $S'(r) = -r$ because $(r, r^{\perp} \oplus W^{\perp}) = 0$. Now $S = i(h)$ for some $h \in H \subset G$. Since h is the identity on $r^{\perp} \oplus W^{\perp}$ because the elements of H are the identity on $W^{\perp}$ and because S is the identity on $r^{\perp}$, and since $h(r) = -r$ because $S(r) = -r$, it follows that $h = S'$; therefore, r is a root of G, as claimed. We can write

$$r_i = ar + br_j$$

where $a \geq 0$ and $b \geq 0$. We cannot have $a = 0$; otherwise, $r_i = br_j$, by lengths and positivity $b = 1$, and $r_i = r_j$, a contradiction. Similarly, by our assumption about $\{r_i, r_j\}$ not being a base for any t, we cannot have $b = 0$. Solving, we get

$$r = (1/a)r_i - br_j.$$

This is a contradiction since r is a root of G and must be positive or negative. Hence, t exists. We are now in the situation of the last lemma. We have

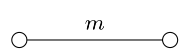
$$\begin{aligned}
\frac{(r_i, r_j)}{\|r_i\|\|r_j\|} &= \cos(\theta) \\
&= \cos(\pi - \varphi) \\
&= -\cos(-\varphi) \\
&= -\cos(\varphi) \\
&= -\cos(2\pi/2m) \\
&= -\cos(\pi/m).
\end{aligned}$$

To complete the proof we note that by the lemma $S_i S_j$ has order m. $\square$

We note that if $P_{ij} = 2$, or equivalently, $(r_i, r_j) = 0$, then S_i and S_j commute:

$$\begin{aligned}
S_i S_j S_i S_j &= 1 \\
S_j S_i &= S_i S_j.
\end{aligned}$$

Next, to every Coxeter group and choice of base $\{r_1, \ldots, r_n\}$ we associate a graph. The graph has n nodes. If $i \neq j$ and $(r_i, r_j) \neq 0$, then we join the i-th and the j-th nodes by a branch and label the branch with p_{ij}; note that as $(r_i, r_j) \neq 0$, we have $p_{ij} \geq 3$. For example, the graph of H_2^m is:



A graph of this type is called a **Coxeter graph**. More precisely, a Coxeter graph is a graph in which edges are labeled with integers greater than or equal to 3; if no label is written, then our convention will be that the label for that edge is 3. The next proposition shows that the Coxeter graph of a Coxeter group classifies the group up to conjugacy by elements of $\mathrm{O}(V)$.

Proposition 8.4. *If G_1 and G_2 in $\mathrm{O}(V)$ are Coxeter groups which have the same Coxeter graph, then there exists $T \in \mathrm{O}(V)$ such that $TG_1T^{-1} = G_2$.*

Proof. Let Π_1 and Π_2 be the bases of G_1 and G_2, respectively. By hypothesis, there exists a bijection

$$T : \Pi_1 \to \Pi_2$$

such that T preserves the labeling of the edges; by a lemma from above, this implies

$$(Tr_i, Tr_j) = (r_i, r_j)$$

for $1 \leq i, j, \leq n$. Let T also denote the linear transformation $T : V \to V$ which sends r_i to Tr_i. Evidently, $T \in \mathrm{O}(V)$. Let $r_i \in \Pi_1$, and let S_{r_i} be the simple reflection corresponding to r_i. Then

$$TS_{r_i}T^{-1} = S_{Tr_i}$$

This implies that $TS_{r_i}T^{-1}$ is contained in G_2. As G_1 is generated by simple reflections, we get $TG_1T^{-1} \subset G_2$; in fact we have $TG_1T^{-1} = G_2$ because all the simple reflections for G_2 are obtained in this way. □

The concept of an irreducible Coxeter group can be characterized in terms of its graph. We say that a Coxeter graph is **connected** if one can get from one node to another node via a path along branches.

Proposition 8.5. *The Coxeter graph of G is connected if and only if G is irreducible.*

Proof. Assume the Coxeter graph of G is connected; and assume G is reducible. Then we can write $\Pi = \Pi_1 \cup \Pi_2$ with $\Pi_1 \perp \Pi_2$ and Π_1 and Π_2 nonempty. As the graph is connected, there exist $a \in \Pi_1$ and $b \in \Pi_2$ and a branch between a and b. This implies that $(a, b) \neq 0$; this is a contradiction. Next, assume G is irreducible; suppose the graph for G is not connected. Since the graph is not connected, we can write

$$\Pi = \Pi_1 \sqcup \Pi_2 \sqcup \cdots \sqcup \Pi_t$$

where any two nodes in Π_i have a path joining them, and there is no path joining an element of Π_i with an element of Π_j for $i \neq j$. Evidently, we have $\Pi_i \perp \Pi_j$ for $i \neq j$. Hence,

$$\Pi = \Pi_1 \cup \Pi_2'$$

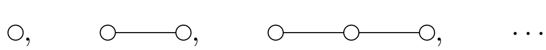
with

$$\Pi_2' = \Pi_2 \cup \cdots \cup \Pi_t$$

and $\Pi_1 \perp \Pi_2'$. This contradicts the assumption that G is irreducible. □

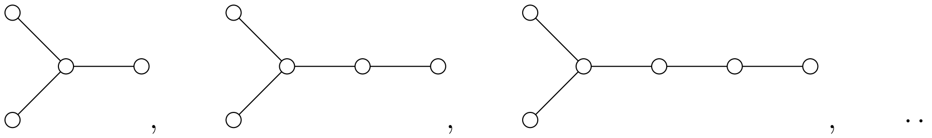
It turns out that the Coxeter graphs of the irreducible Coxeter groups are as follows. The subscript is the number of roots in the base, or equivalently, the dimension of the space that the Coxeter group acts on.

A_n, $n \geq 1$:

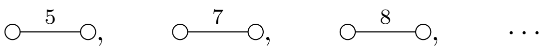


B_n, $n \geq 2$:

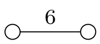
D_n, $n \geq 4$



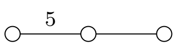
H_2^n, $n \geq 5$, $n \neq 6$:



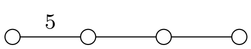
G_2:



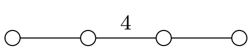
I_3:



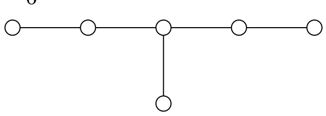
I_4:



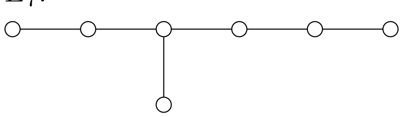
F_4:



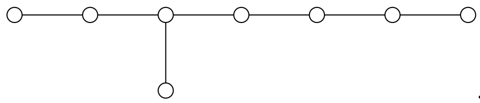
E_6:



E_7:



E_8:



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Before proving that these are indeed all the Coxeter graphs of the irreducible Coxeter groups we will make few observations based on this fact. First of all, it is evident that mostly the angle between r_i and r_j is

$$\frac{3-1}{3}\pi = \frac{2}{3}\pi = 120 \text{ degrees.}$$

Equivalently, it means that mostly S_iS_j has order three. Sometimes, it can happen that the angles and orders of S_iS_j are

$$\begin{aligned}
\frac{4-1}{4}\pi &= \frac{3}{4}\pi = 135 \text{ degrees},\ S_iS_j \text{ order } 4\\
\frac{5-1}{5}\pi &= \frac{4}{5}\pi = 144 \text{ degrees},\ S_iS_j \text{ order } 5\\
\frac{6-1}{6}\pi &= \frac{5}{6}\pi = 150 \text{ degrees},\ S_iS_j \text{ order } 6
\end{aligned}$$

but the angles and orders don't get any bigger except for the dihedral group H_2^n. We can make a list of the irreducible Coxeter groups in low dimensions:

Dimension 1 : A_1 (2 elements);
Dimension 2 : A_2 ($\cong H_2^3$)(6 elements), B_2 ($\cong H_2^4$) (8 elements),
G_2 ($\cong H_2^6$) (12 elements), $H_2^n, n \geq 5, n \neq 6$ ($2n$ elements);
Dimension 3 : A_3 (24 elements), B_3 (48 elements), I_3 (120 elements).

To prove that the above is the list of all the Coxeter graphs of all the irreducible Coxeter groups we will need to introduce another property that these particular Coxeter graphs enjoy. Suppose we are given a Coxeter graph with m nodes. Number the nodes. Then to this graph we can associate an $m \times m$ symmetric matrix (α_{ij}) of real numbers by setting

$$\begin{aligned}
\alpha_{ii} &= 1,\\
\alpha_{ij} &= -\cos(\pi/p_{ij}) \text{ if the } i\text{-th and } j\text{-th node are joined by a branch labeled } p_{ij},\\
\alpha_{ij} &= 0 \text{ otherwise.}
\end{aligned}$$

The matrix (α_{ij}) is a symmetric matrix. We can associate a quadratic form on $\mathbb{R}^m$ to (α_{ij}); it is defined by

$$Q(\lambda_1, \ldots, \lambda_m) = \sum_{i,j} \alpha_{ij}\lambda_i\lambda_j.$$

The equivalence class of this quadratic form does not depend on the choice of numbering of the nodes.

Proposition 8.6. *The Coxeter graph of a Coxeter group is positive definite.*

Proof. As usual, let $\{r_1, \ldots, r_n\}$ be the base of the root system. Then

$$\alpha_{ij} = (r_i, r_j).$$

Therefore,

$$\begin{aligned} Q(\lambda_1, \ldots, \lambda_n) &= \sum_{i,j} (r_i, r_j)\lambda_i\lambda_j \\ &= (\sum_i, \lambda_i r_i, \sum_j \lambda_j r_j) \\ &= \|\lambda_1 r_1 + \cdots + \lambda_n r_n\|^2 \\ &> 0, \end{aligned}$$

if $(\lambda_1, \ldots, \lambda_n) \neq 0$. $\square$

Proposition 8.7. *The Coxeter graphs listed above are connected and positive definite.*

Proof. It is clear from inspection that these Coxeter graphs are connected. To prove that they are positive definite we will use the fact that a matrix $B = (B_{ij}), 1 \leq i, j \leq n$ is positive definite if and only if $\det B(k) > 0$ for $1 \leq k \leq n$, where $B(k) = (B_{ij}), 1 \leq i, j \leq k$. Let us consider the matrix of the Coxeter graph A_n, which we will also call A_n. This matrix has the form:

$$\begin{bmatrix} 1 & -\cos(\pi/3) & 0 & 0 & \ldots \\ -\cos(\pi/3) & 1 & -\cos(\pi/3) & 0 & \ldots \\ 0 & -\cos(\pi/3) & 1 & -\cos(\pi/3) & \ldots \\ 0 & 0 & -\cos(\pi/3) & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}.$$

This is:

$$\begin{bmatrix} 1 & -1/2 & 0 & 0 & \ldots \\ -1/2 & 1 & -1/2 & 0 & \ldots \\ 0 & -1/2 & 1 & -1/2 & \ldots \\ 0 & 0 & -1/2 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}.$$

We see that the matrix A_{n-1} is obtained by deleting the last row and column of A_n; also, we see that $A_n(n-1) = A_{n-1}$. So it will suffice to show that $\det A_n > 0$ for all n. Expanding on the last row, we have

$$\det A_n = \sum_{j=1}^{n} (-1)^{n+j} A_{nj} \det A(n|j)$$

$$\begin{aligned} &= (-1)^{2n-1} A_{n,n-1} \det A(n|n-1) + (-1)^{n+n} A_{n,n} \det A(n|n) \\ &= -(-1/2)(-1/2) \det A_{n-2} + \det A_{n-1} \\ \det A_n &= -(1/4) \det A_{n-2} + \det A_{n-1}. \end{aligned}$$

This is valid for $n \geq 3$. We also have

$$\det A_1 = 1, \quad \det A_2 = 3/4.$$

One can now prove by induction that

$$\det A_n = (n+1)/2^n.$$

This is nonzero, so A_n is positive definite. We have

$$B_n = \begin{bmatrix} 1 & -\cos(\pi/4) & 0 & 0 & 0 & \dots \\ -\cos(\pi/4) & 1 & -\cos(\pi/2) & 0 & 0 & \dots \\ 0 & -\cos(\pi/2) & 1 & -\cos(\pi/2) & 0 & \dots \\ 0 & 0 & -\cos(\pi/2) & 1 & -\cos(\pi/2) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix},$$

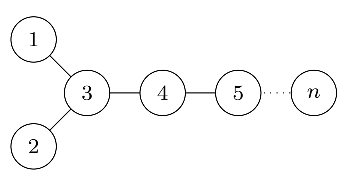
which is

$$B_n = \begin{bmatrix} 1 & -\sqrt{2}/2 & 0 & 0 & 0 & \dots \\ -\sqrt{2}/2 & 1 & -1/2 & 0 & 0 & \dots \\ 0 & -1/2 & 1 & -1/2 & 0 & \dots \\ 0 & 0 & -1/2 & 1 & -1/2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

We can compute the determinant of this by expanding on the first row:

$$\begin{aligned} \det B_n &= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det B(1|j) \\ &= (-1)^{1+1} B_{11} \det B(1|1) + (-1)^{1+2} B_{12} \det B(1|2) \\ &= \det A_{n-1} - (-\sqrt{2}/2)(-\sqrt{2}/2) \det A_{n-2} \\ &= \det A_{n-1} - (1/2) \det A_{n-2} \\ &= (n-1+1)/2^{n-1} - (1/2)(n-2+1)/2^{n-2} \\ &= 1/2^{n-1}. \end{aligned}$$

Next, to deal with D_n, we need to choose a labeling:



We have then

$$D_n = \begin{bmatrix} 1 & 0 & -\cos(\pi/3) & 0 & 0 & 0 & \dots \\ 0 & 1 & -\cos(\pi/3) & 0 & 0 & 0 & \dots \\ -\cos(\pi/3) & -\cos(\pi/3) & 1 & -\cos(\pi/3) & 0 & 0 & \dots \\ 0 & 0 & -\cos(\pi/3) & 1 & -\cos(\pi/3) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

which is

$$D_n = \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1/2 & 0 & 0 & 0 & \dots \\ -1/2 & -1/2 & 1 & -1/2 & 0 & 0 & \dots \\ 0 & 0 & -1/2 & 1 & -1/2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Expanding on the first row, we get:

$$\begin{aligned} \det D_n &= \sum_{j=1}^{n} (-1)^{1+j} D_{1j} \det D(1|j) \\ &= (-1)^{1+1} D_{11} \det D(1|1) + (-1)^{1+3} D_{13} \det D(1|3) \\ &= \det A_{n-1} + (-1/2) \det D(1|3), \end{aligned}$$

where

$$D(1|3) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ -1/2 & -1/2 & -1/2 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1/2 & 0 & \dots \\ 0 & 0 & -1/2 & 1 & -1/2 & \\ \dots & \dots & \dots & \dots & \dots & \end{bmatrix}.$$

We have

$$\det D(1|3) = \det \begin{bmatrix} 0 & 1 \\ -1/2 & -1/2 \end{bmatrix} \cdot \det A_{n-3} = (1/2) \det A_{n-3}.$$

Hence,

$$\det D_n = (n-1+1)/2^{n-1} + (-1/2)(1/2)(n-3+1)/2^{n-3} = 1/2^{n-2}.$$

We have

$$H_2^m = \begin{bmatrix} 1 & -\cos(\pi/m) \\ -\cos(\pi/m) & 1 \end{bmatrix}.$$

Hence,

$$\det H_2^m = 1 - \cos^2(\pi/m) = \sin^2(\pi/m) > 0$$

as $m \geq 5$. Turning to I_3, we see that $\det I_3(2) = \det H_2^5 > 0$. Concerning $\det I_3$, we have

$$I_3 = \begin{bmatrix} 1 & -\cos(\pi/5) & 0 \\ -\cos(\pi/5) & 1 & -\cos(\pi/3) \\ 0 & -\cos(\pi/3) & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\cos(\pi/5) & 0 \\ -\cos(\pi/5) & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix}.$$

The determinant of this matrix is:

$$\det I_3 = 3/4 - \cos^2(\pi/5) = 3/4 - [(1+\sqrt{5})/4]^2 = (3-\sqrt{5})/8 > 0.$$

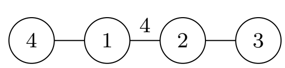
Hence, I_3 is positive definite. To check that I_4 is positive definite we just need to check that $\det I_4 > 0$. We have

$$\begin{aligned} I_4 &= \begin{bmatrix} 1 & -\cos(\pi/5) & 0 & 0 \\ -\cos(\pi/5) & 1 & -\cos(\pi/3) & 0 \\ 0 & -\cos(\pi/3) & 1 & -\cos(\pi/3) \\ 0 & 0 & -\cos(\pi/3) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\cos(\pi/5) & 0 & 0 \\ -\cos(\pi/5) & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}. \end{aligned}$$

We get

$$\det I_4 = 1/2 - (3/4)\cos^2(\pi/5) = (7-3\sqrt{5})/32 > 0.$$

If we label F_4 in the following way



then we have

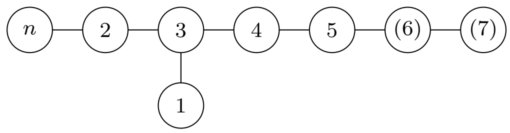
$$F_4 = \left[\begin{array}{ccc|c} & & & -1/2 \\ & B_3 & & 0 \\ & & & 0 \\ -1/2 & 0 & 0 & 1 \end{array}\right].$$

So all we need to check is that this matrix has positive determinant. We get

$$\begin{aligned} \det F_4 &= \det \left[\begin{array}{cccc} & & & -1/2 \\ & B_3 & & 0 \\ & & & 0 \\ -1/2 & 0 & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{cc} B_3 + \begin{bmatrix} -1/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{matrix} -1/2 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & 1 \end{array}\right] \end{aligned}$$

$$\begin{aligned}
&= \det \begin{bmatrix} -1/4 & * \\ 0 & A_2 \end{bmatrix} + \det B_3 \\
&= -(1/4) \det A_2 + \det B_3 \\
&= 1/2^4.
\end{aligned}$$

Finally, we consider E_6, E_7 and E_8. We label these as



With this labeling, because we have already treated the case of D_{n-1} it suffices to check that $\det E_n > 0$. We have

$$E_n = \begin{bmatrix} & & & & & 0 \\ & & & & & -1/2 \\ & & D_{n-1} & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ 0 & -1/2 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

We calculate the determinant of this as in the F_4 case. This yields

$$\begin{aligned}
\det E_n &= \det D_{n-1} + \det \begin{bmatrix} 1 & 0 & -1/2 & 0 & \dots \\ 0 & -1/4 & -1/2 & 0 & \dots \\ -1/2 & 0 & 1 & -1/2 & \dots \\ 0 & 0 & -1/2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\
&= \det D_{n-1} - \det \begin{bmatrix} 0 & 1 & -1/2 & 0 & \dots \\ -1/4 & 1 & -1/2 & 0 & \dots \\ 0 & -1/2 & 1 & -1/2 & \dots \\ 0 & 0 & -1/2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\
&= \det D_{n-1} + \det \begin{bmatrix} -1/4 & 1 & -1/2 & 0 & \dots \\ 0 & 1 & -1/2 & 0 & \dots \\ 0 & -1/2 & 1 & -1/2 & \dots \\ 0 & 0 & -1/2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\
&= \det D_{n-1} - (1/4) \det A_{n-2} \\
&= (9-n)/2^n
\end{aligned}$$

$$> \quad 0.$$

This completes the proof. □

We consider a concept that is more general than a Coxeter graph. A **marked graph** is a graph in which each branch is labeled with a number $p_{ij} > 2$. As before, we can associate to a marked graph a symmetric matrix. A Coxeter graph is a marked graph in which each label is an integer. If H is a marked graph, then we say that H is a **subgraph of a marked graph** G if H can be obtained from G by removing some of the nodes of G and adjoining branches, or by decreasing the marks on some of the branches of G.

Lemma 8.8. *A nonempty subgraph H of a positive definite marked graph G is also positive definite.*

Proof. Let the nodes of G be $a_1, \ldots, a_m$, with $a_1, \ldots, a_k$ the nodes of H. Let the labels for the branches of G and H be p_{ij} and q_{ij}, respectively. Let the matrices associated to G and H be $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, respectively. Let the quadratic forms associated to G and H be Q_G and Q_H, respectively. We need to show that Q_H is positive definite, i.e, $Q_H(x) > 0$ for $x \in \mathbb{R}^k$, $x \neq 0$. Suppose that $Q_H(x) \leq 0$ for some $x \neq 0$. We have

$$q_{ij} \leq p_{ij}, \quad 1 \leq i, j \leq k.$$

Hence,

$$\begin{aligned}
q_{ij} &\leq p_{ij} \\
1/q_{ij} &\geq 1/p_{ij} \\
\pi/q_{ij} &\geq \pi/p_{ij} \\
\cos(\pi/q_{ij}) &\leq \cos(\pi/p_{ij}) \\
-\cos(\pi/q_{ij}) &\geq -\cos(\pi/p_{ij}) \\
\beta_{ij} &\geq \alpha_{ij},
\end{aligned}$$

for $1 \leq i, j \leq k$. Write $x = (\lambda_1, \ldots, \lambda_k)$ and define $y = (|\lambda_1|, \ldots, |\lambda_k|, 0, \ldots, 0) \in \mathbb{R}^m$. Then using the fact that $\beta_{ij} \leq 0$ for $i \neq j$, and the above inequality,

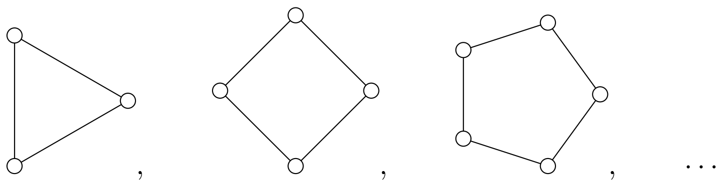
$$\begin{aligned}
0 &\geq Q_H(x) \\
&= \sum_{i,j} \beta_{ij} \lambda_i \lambda_j \\
&\geq \sum_{i,j} \beta_{ij} |\lambda_i||\lambda_j| \\
&\geq \sum_{i,j} \alpha_{ij} |\lambda_i||\lambda_j| \\
&= Q_G(y)
\end{aligned}$$

$$> \quad 0,$$

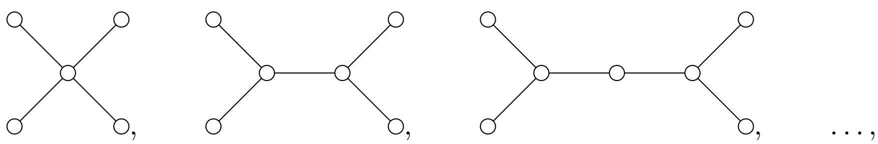
which is a contradiction. $\square$

We will also need a list of certain marked graphs which are not positive definite, and which in fact have determinant zero:

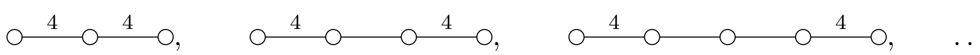
P_n, $n \geq 3$:



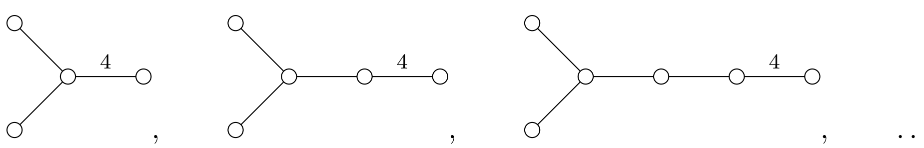
Q_n, $n \geq 5$:



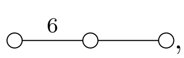
S_n, $n \geq 3$:



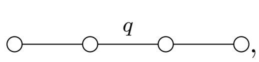
T_n, $n \geq 4$:



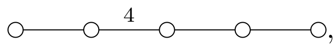
U_3:



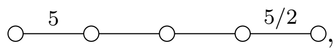
Z_4, $\cos(\frac{\pi}{q}) = \frac{3}{4}$, $4 < q < 5$:



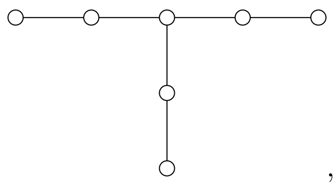
V_5:



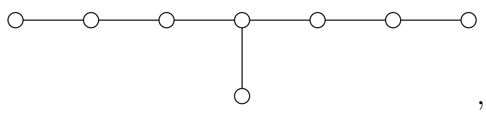
Y_5:



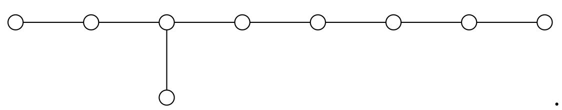
R_7:



R_8:



R_9:



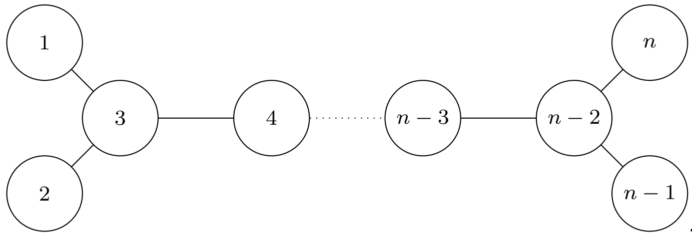
Lemma 8.9. *The above graphs are not positive definite.*

Proof. The matrix of P_n has the form:

$$\begin{bmatrix} 1 & -1/2 & 0 & \dots & 0 & -1/2 \\ -1/2 & 1 & -1/2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -1/2 \\ -1/2 & 0 & 0 & \dots & -1/2 & 1 \end{bmatrix}.$$

If we add to the first row the other rows then the resulting matrix has first row consisting of just zeros. Hence, $\det P_n = 0$.

To calculate the determinant of Q_n we label as follows



.

Then the matrix of Q_n is:

$$\begin{bmatrix} & & & 0 & 0 & 0 \\ & D_{n-3} & & \vdots & \vdots & \vdots \\ & & & -1/2 & 0 & 0 \\ 0 & \dots & -1/2 & 1 & -1/2 & -1/2 \\ 0 & \dots & 0 & -1/2 & 1 & 0 \\ 0 & \dots & 0 & -1/2 & 0 & 1 \end{bmatrix}$$

Expanding on the last row:

$$\begin{aligned}
\det Q_n &= \sum_{j=1}(-1)^{n+j} Q_{nj} \det Q(n|j) \\
&= (-1/2)\det \begin{bmatrix} & & & 0 & 0 \\ & D_{n-3} & & \vdots & \vdots \\ & & & -1/2 & 0 \\ 0 & \dots & -1/2 & -1/2 & -1/2 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} + 1 \cdot \det D_{n-1} \\
&= (-1/2)\det \begin{bmatrix} & & & 0 & 0 \\ & D_{n-3} & & \vdots & \vdots \\ & & & -1/2 & 0 \\ 0 & \dots & 0 & -1/2 & -1/2 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} + \det D_{n-1} \\
&= \det D_{n-1} - (1/4)\det D_{n-3} \\
&= 1/2^{n-1-2} - (1/2^2)(1/2^{n-3-2} \\
&= 0.
\end{aligned}$$

The remaining determinants may be shown to be zero by similar methods. □

Theorem 8.10. *If G is a connected positive definite Coxeter graph, then G is one of the graphs A_n, B_n, D_n, H_2^n, G_2, I_3, I_4, F_4, E_6, E_7 or E_8.*

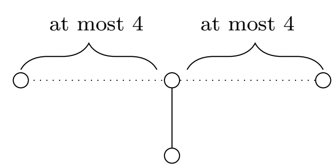
Proof. First of all, we see that since P_n is never positive definite, G does not contain a circuit.

Next, we consider the possible labels of G. Suppose G has a label n with $n \geq 6$. Then H_2^n for $n \geq 7$ or G_2 is contained in G. In fact, we get $G = H_2^n$ or $G = G_2$, for otherwise U_3 is contained in G. We may therefore assume that every label is $3, 4$ or 5.

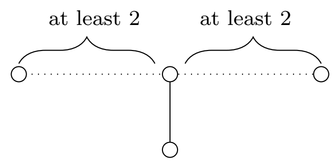
The remainder of the argument is now divided into cases. Suppose first that 4 is a label. Then since S_n is not a subgraph of G, 4 is a label for exactly one edge. Also, G cannot have a branch vertex: otherwise, T_n is a subgraph. Thus, G is just a sequence of vertices and edges, with one edge labeled 4. As S_n is not a subgraph, no other edge can be labeled 5. So all the edges are labeled 3, except for one edge which is labeled 4. If the edge labeled 4 is one of the end edges, G is a B_n. Assume the edge labeled 4 is not an end edge. If there are at least two edges on one side of the edge labeled 4, then V_5 is a subgraph, which is impossible. Hence, G is F_4.

Next, suppose 4 is not a label, so that the possible labels are 3 and 5. Suppose that 5 is a label. Then 5 is a label for exactly one edge: otherwise, S_n is a subgraph. Also, again G cannot have a branch point: otherwise T_n is a subgraph. So G is a just a sequence of vertices and edges, with one edge labeled 5 and the rest labeled 3. The edge labeled 5 cannot be an interior edge: otherwise, Z_4 is a subgraph. So the edge labeled 5 is an end edge. There can be at most two edges besides the end edge labeled 5: otherwise, Y_5 is s subgraph. Therefore, G is H_2^5, I_3 or I_4.

We now must deal with the remaining case when all the edges are labeled 3. Since Q_n is not a subgraph, G can have at most one branch vertex; moreover, that branch vertex, if it exists, must have exactly three edges coming out of it. If G has no branch vertex, then G is an A_n; assume G has a (unique) branch vertex. Then at least one of the edges coming out of the branch vertex is not connected to another edge: otherwise R_7 is a subgraph. If another edge coming out of the branch vertex is also not connected to another edge, then G is a D_n. Assume, therefore, that exactly one edge coming out of the branch vertex is connected to no other edge. Then besides the single edge coming out of the branch vertex, the two other sequences of edges can be of length at most four: otherwise, R_9 is a subgraph. The picture is:



We cannot have at least three edges and at least three edges for otherwise R_8 is a subgraph. So we must have



and at most one has three or four. Hence, G is E_6, E_7 or E_8. $\square$

9 The crystallographic condition

We say that a Coxeter group G is **crystallographic** if there exists a lattice in $\mathbb{R}^n$ which is stabilized by G. We will now determine all the possible crystallographic irreducible Coxeter groups.

Lemma 9.1. *If G is a crystallographic Coxeter group, then all of the p_{ij} are $1, 2, 3, 4$ or 6.*

Proof. Suppose G is a crystallographic Coxeter group. Let $i \neq j$. To prove that $p_{ij} = 2, 3, 4$ or 6 we will compute the trace of $S_i S_j$ in two different ways. Since G stabilizes a lattice, there exists a basis for V such that all the matrices in G have integral entries. Hence, the trace of $S_i S_j$ is an integer.

On the other hand, we saw before that there was a basis for V such that the matrix of $S_i S_j$ in this basis has the form

$$\begin{bmatrix} A & 0 \\ 0 & I_{n-2} \end{bmatrix}$$

where

$$A = \begin{bmatrix} \cos(2\pi/p_{ij}) & -\sin(2\pi/p_{ij}) \\ \sin(2\pi/p_{ij}) & \cos(2\pi/p_{ij}) \end{bmatrix}.$$

It follows that the trace of $S_i S_j$ is

$$2\cos(2\pi/p_{ij}) + (n-2).$$

This is an integer if and only $2\cos(2\pi/p_{ij})$ is an integer. We have:

$$\begin{aligned} 2\cos(2\pi/2) &= 0, \\ 2\cos(2\pi/3) &= 1, \\ 2\cos(2\pi/4) &= 0, \\ 2\cos(2\pi/5) &= (-1+\sqrt{5})/2, \\ 2\cos(2\pi/6) &= 1. \end{aligned}$$

And

$$\begin{aligned} p_{ij} &> 6 \\ 2\pi/p_{ij} &< \pi/3 \\ 2 > 2\cos(2\pi/p_{ij}) &> 2\cos(\pi/3) = 1. \end{aligned}$$

Hence, if $p_{ij} > 6$, then $2\cos(2\pi/p_{ij})$ is not an integer. $\square$

Corollary 9.2. *If G is an irreducible Coxeter group and G is crystallographic, then G has Coxeter graph A_n, B_n, D_n, G_2, F_4, E_6, E_7 or E_8.*

Next, suppose G is an irreducible Coxeter group and G has Coxeter graph one of the graphs listed in the last corollary: is G crystallographic? We will show it is. To do so, we must produce a lattice which is invariant under all the elements of G. A natural object to look at is the base $\Pi = \{r_1, \dots, r_n\}$ for G. This is a basis for V, and we can consider the lattice spanned by these n linearly independent vectors:

$$\mathbb{Z}r_1 + \cdots + \mathbb{Z}r_n.$$

It turns out that G often stabilizes this lattice – but not always. To get a lattice stabilized by G we need to modify the lengths of the r_i. We do this as follows:

$$\{r_1', \dots, r_n'\} = \begin{cases} \{r_1, \dots, r_n\} & \text{if } G = A_n, D_n, E_6, E_7, E_8 \\ \{(1/\sqrt{2})r_1, r_2, \dots, r_n\} & \text{if } G = B_n \\ \{(1/\sqrt{3})r_1, r_2\} & \text{if } G = G_2 \\ \{r_1, r_2, (1/\sqrt{2})r_3, (1/\sqrt{2})r_4\} & \text{if } G = F_4 \end{cases}$$

Lemma 9.3. *If G is an irreducible Coxeter group with Coxeter graph A_n, B_n, D_n, G_2, F_4, E_6, E_7 or E_8, then*

$$\begin{aligned} p_{ij} = 3 \quad &\Longrightarrow \quad \|r_i'\| = \|r_j'\| \\ p_{ij} = 4 \quad &\Longrightarrow \quad \|r_i'\| = \sqrt{2}\|r_j'\| \ \text{or}\ \|r_j'\| = \sqrt{2}\|r_i'\| \\ p_{ij} = 6 \quad &\Longrightarrow \quad \|r_i'\| = \sqrt{3}\|r_j'\| \ \text{or}\ \|r_j'\| = \sqrt{3}\|r_i'\|. \end{aligned}$$

Proof. This follows by inspection; it is useful to look at the Coxeter graph. □

Theorem 9.4. *If G is a Coxeter group with Coxeter graph A_n, B_n, D_n, G_2, F_4, E_6, E_7 or E_8, then G is crystallographic.*

Proof. We define the following lattice:

$$L = \mathbb{Z}r_1' + \cdots + \mathbb{Z}r_n'.$$

We need to verify that if $1 \le i, j \le n$ then $S_i r_j' \in L$. There are several cases. If $p_{ij} = 1$, i.e., $i = j$, then $S_i r_j' = S_i r_i' = -r_i' \in L$. If $p_{ij} = 2$, then $(r_i', r_j') = 0$. Hence,

$$S_i r_j' = r_j' - 2\frac{(r_j', r_i')}{(r_i', r_i')} r_i' = r_j' \in L.$$

Assume $p_{ij} = 3$. Then

$$S_i r_j' = r_j' - 2\frac{(r_j', r_i')}{(r_i', r_i')} r_i'.$$

We also have

$$\frac{(r_i', r_j')}{\|r_i'\|\|r_j'\|} = \frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = -\cos(\pi/3) = -1/2,$$

so that
$$(r_i', r_j') = (-1/2)\|r_i'\|\|r_j'\|$$
Substituting, we get:
$$S_i r_j' = r_j' - 2\frac{(-1/2)\|r_i'\|\|r_j'\|}{(r_i', r_i')} r_i' = r_j' - 2\frac{(-1/2)\|r_i'\|^2}{\|r_i'\|^2} r_i' = r_j' + r_i' \in L.$$
Suppose $p_{ij} = 4$. Then
$$\frac{(r_i', r_j')}{\|r_i'\|\|r_j'\|} = \frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = -\cos(\pi/4) = -1/\sqrt{2},$$
so that
$$S_i r_j' = r_j' - 2\frac{(-1/\sqrt{2})\|r_i'\|\|r_j'\|}{(r_i', r_i')} r_i' = r_j' - 2\frac{(-1/\sqrt{2})\|r_i'\|\|r_j'\|}{\|r_i'\|^2} r_i' = r_j' + \sqrt{2}\frac{\|r_j'\|}{\|r_i'\|} r_i'.$$
If $\|r_i'\| = \sqrt{2}\|r_j'\|$, then we have
$$S_i r_j' = r_j' + \sqrt{2}\frac{\|r_j'\|}{\|r_i'\|} r_i' = r_j' + r_i' \in L.$$
If $\|r_j'\| = \sqrt{2}\|r_i'\|$, then we have
$$S_i r_j' = r_j' + \sqrt{2}\frac{\|r_j'\|}{\|r_i'\|} r_i' = r_j' + 2r_i' \in L.$$
Suppose $p_{ij} = 6$. Then
$$\frac{(r_i', r_j')}{\|r_i'\|\|r_j'\|} = \frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = -\cos(\pi/6) = -\frac{\sqrt{3}}{2},$$
so that
$$S_i r_j' = r_j' - 2\frac{(-\sqrt{3}/2)\|r_i'\|\|r_j'\|}{(r_i', r_i')} r_i' = r_j' - 2\frac{(-\sqrt{3}/2)\|r_i'\|\|r_j'\|}{\|r_i'\|^2} r_i' = r_j' + \sqrt{3}\frac{\|r_j'\|}{\|r_i'\|} r_i'.$$
Suppose $\|r_i'\| = \sqrt{3}\|r_j'\|$. Then
$$S_i r_j' = r_j' + \sqrt{3}\frac{\|r_j'\|}{\|r_i'\|} r_i' = r_j' + r_i' \in L.$$
Suppose $\|r_j'\| = \sqrt{3}\|r_i'\|$. Then
$$S_i r_j' = r_j' + \sqrt{3}\frac{\|r_j'\|}{\|r_i'\|} r_i' = r_j' + 3r_i' \in L.$$
This completes the proof. $\square$

10 Existence of Coxeter groups

Theorem 10.1. *There exist Coxeter groups with Coxeter graphs*

$$A_n, B_n, D_n, H_2^n, G_2, I_3, I_4, F_4, E_6, E_7, E_8.$$

Proof. Construction of A_n: Let $e_1, \ldots, e_{n+1}$ be the standard basis vectors for $\mathbb{R}^{n+1}$. We can let the symmetric group S_{n+1} act on $\mathbb{R}^{n+1}$ by defining

$$Te_i = e_{T(i)}$$

and extending by linearity. Consider the transpositions

$$S_1 = (e_1 e_2), S_2 = (e_2 e_3), \ldots, S_n = (e_n e_{n+1}).$$

These elements generate S_{n+1}. We claim that they are also reflections. To see this, let $1 \leq i \leq n$. Set

$$r_i = e_{i+1} - e_i.$$

Then

$$r_i^\perp = \mathbb{R}e_1 + \ldots \mathbb{R}e_{i-1} + \mathbb{R}(e_i + e_{i+1}) + \mathbb{R}e_{i+2} + \ldots \mathbb{R}e_{n+1}.$$

And we have

$$\begin{aligned} S_i e_1 &= e_1 \\ &\ldots \\ S_i e_{i-1} &= e_{i-1} \\ S_i(e_i + e_{i+1}) &= e_i + e_{i+1} \\ S_i e_{i+2} &= e_{i+2} \\ &\ldots \\ S_i e_{n+1} &= e_{n+1}. \end{aligned}$$

Also,

$$S_i r_i = -r_i.$$

This proves that S_i is a reflection, and that r_i is a root for S_i; strictly speaking, we should normalize r_i so that it has length one, but we will not do this explicitly. Since S_{n+1} is generated by transpositions, we conclude that S_{n+1} is a reflection group. However, S_{n+1} is not a Coxeter group: it is not effective. The vector

$$(1, \ldots, 1)$$

is stabilized by all the elements of S_{n+1}.

To obtain a Coxeter group we define

$$V = (1, \ldots, 1)^\perp = \{(x_1, \ldots, x_{n+1}) : x_1 + \cdots + x_{n+1} = 0\}.$$

This hyperplane is preserved by the elements of S_{n+1}. We denote by A_n the subgroup obtained by restricting the elements of S_{n+1} to V: this is a subgroup of $\mathrm{O}(V)$. Of course, the restriction of elements of S_{n+1} to V is injective. The group A_n is generated by reflections, and is thus a reflection group. It is also effective. Assume $v = a_1e_1 + \cdots + a_{n+1}e_{n+1} \in V$ is fixed by all the elements of A_n. Then for $1 \leq i \leq n$,

$$\begin{aligned} v &= S_i v \\ &= a_1e_1 + \ldots a_{i-1}e_{i-1} + a_{i+1}e_i + a_ie_{i+1} + a_{i+2}e_{i+2} + \cdots + a_{n+1}e_{n+1}. \end{aligned}$$

This implies that $a_1 = \cdots = a_{n+1}$. Hence, $v \in V^\perp$, and so $v = 0$. It follows that A_n is a Coxeter group.

Next, we compute the root system of A_n. Already, we know that $r_1, \ldots, r_n$ are roots of A_n. Since these are roots for a generating set of A_n, to find the root system of A_n it suffices to apply all the elements of A_n to $r_1, \ldots, r_n$. This is evidently:

$$\Delta = \{e_i - e_j : i \neq j, 1 \leq i, j \leq n+1\}.$$

We claim that there exists an element $t \in V$ such that $(t, r_1) > 0, \ldots, (t, r_n) > 0$, i.e., that the r_i all lie on the same side of a half-plane. Let $t = a_1e_1 + \cdots + a_{n+1}e_{n+1}$ be an element of V. Then:

$$(t, r_i) = a_{i+1} - a_i.$$

We thus need

$$a_1 < \cdots < a_{n+1} \quad \text{and} \quad a_1 + \cdots + a_{n+1} = 0.$$

The choice

$$a_1 = -(2 + \cdots + n + 1), a_2 = 2, \ldots, a_{n+1} = n + 1.$$

We use t to define positive and negative roots. We get:

$$(t, e_i - e_j) = a_i - a_j.$$

This implies that

$$\begin{aligned} \Delta^+ &= \{e_i - e_j : i > j, 1 \leq i, j \leq n+1\}, \\ \Delta^- &= \{e_i - e_j : i < j, 1 \leq i, j, \leq n+1\}. \end{aligned}$$

The roots $r_1, \ldots, r_n$ are clearly positive. Moreover, $\{r_1, \ldots, r_n\}$ forms a base for this root system. If $e_i - e_j$, $i > j$, is an arbitrary positive root, then

$$e_i - e_j = r_{i-1} + r_{i-2} + \cdots + r_j.$$

Hence,

$$\Pi = \{r_1, \ldots, r_n\}.$$

Finally, we have for $i < j$:

$$\frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = \begin{cases} 0 & \text{if } j \neq i+1 \\ -1/2 = -\cos(\pi/3) & \text{if } j = i+1. \end{cases}$$

This implies that the Coxeter graph of A_n is

as desired.

Next, we construct a Coxeter group with graph B_n. This time, we will work directly inside $\mathrm{O}(\mathbb{R}^n)$, with no need to go to a smaller space inside our ambient space. Again, the symmetric group S_n acts $\mathbb{R}^n$ via permutations of the coordinates. It is generated by the transpositions:

$$S_2 = (e_1 e_2), S_3 = (e_2 e_3), \ldots, S_n = (e_{n-1} e_n).$$

As before, $S_1, \ldots, S_{n-1}$ is contained in $\mathrm{O}(\mathbb{R}^n)$, and in fact these elements are reflections with $S_1, \ldots, S_{n-1}$ having respectively roots

$$r_2 = e_2 - e_1, r_2 = e_3 - e_2, \ldots, r_n = e_n - e_{n-1}.$$

Besides the subgroup $S_{n-1} \subset \mathrm{O}(\mathbb{R}^n)$ we will also consider a subgroup of $\mathrm{O}(\mathbb{R}^n)$. Namely, we consider the elements defined by

$$S_{e_i}(e_j) = \begin{cases} e_j & \text{if } j \neq i \\ -e_i & \text{if } j = i. \end{cases}$$

These elements are defined for $1 \leq i \leq n$, and are clearly contained in $\mathrm{O}(\mathbb{R}^n)$. We let K_n denote the subgroup of $\mathrm{O}(\mathbb{R}^n)$ generated by these elements. Since the elements S_{e_i} all commute with each other, and since each has order two, K_n is the direct product of the subgroups generated by the S_{e_i}, and has order 2^n. For each element k of K_n there exists a unique subset J of $\{e_1, \ldots, e_n\}$ such that

$$k = \prod_{e_i \in J} S_{e_i}.$$

We denote the element corresponding to J by f_J. Of course, it sends e_i to $-e_i$ if and only if $e_i \in J$; otherwise, it sends e_i to e_i. We see that

$$f_J f_L = f_{J\Delta L}$$

where $J\Delta L$ is the symmetric difference $(J - L) \cup (L - J) = (J \cup L) - (J \cap L)$. We claim that S_n normalizes K_n. Let $T \in S_n$ and $f_J \in K_n$. We assert that

$$T f_J T^{-1} = f_{T(J)}.$$

To see this, we compute. Let e_i be a standard basis vector. Then $T^{-1}e_i$ is another standard basis vector, say, $T^{-1}e_i = e_j$. We get

$$T f_J T^{-1} e_i = T f_J e_j.$$

If $e_j \in J$, this is:

$$T f_J T^{-1} e_i = T f_J e_j = T(-e_j) = -T e_j = -e_i.$$

If $e_J \notin J$, this is:

$$T f_J T^{-1} e_i = T f_J e_j = T e_j = e_i.$$

This proves our claim. We also note that

$$K_n \cap S_n = 1.$$

This is clear because the elements of S_n never change the sign of a standard basis vector and the only element of K_n that does not change the sign of some standard basis vector is the identity. We may now consider the semi-direct product

$$B_n = K_n \rtimes S_n.$$

This is the subgroup of $\mathrm{O}(\mathbb{R}^n)$ generated by the subgroups K_n and S_n. Every element of B_n may be written in the form

$$f_J T$$

where $T \in S_n$. We have

$$f_J T f_L S = f_J T f_L T^{-1} T S = f_J f_{T(L)} T S = f_{J\Delta T(L)} T S.$$

What elements generate B_n? We know that B_n is generated by the elements:

$$f_{e_1}, \ldots, f_{e_n}, S_2, \ldots, S_n.$$

But in fact B_n is generated by smaller set. Let $T \in S_n$ send e_1 to e_i; there are many elements that do this. Then

$$T f_{e_1} T^{-1} = f_{\{Te_1\}} = f_{e_i}.$$

Since T is a product of elements from $\{S_2, \ldots, S_n\}$, it follows that B_n is generated by

$$\{f_{e_1}, S_2, \ldots, S_n\}.$$

We already know that S_i has a root $r_i = e_i - e_{i-1}$ for $i = 2, \ldots, n$; the element f_{e_1} has a root $r_1 = e_1$. The set

$$\{r_1 = e_1, r_2 = e_2 - e_1, \ldots, r_n = e_n - e_{n-1}\}$$

is a basis for $\mathbb{R}^n$. Is B_n a Coxeter group? It is a finite subgroup of $\mathrm{O}(\mathbb{R}^n)$ generated by reflections. Moreover, since a basis for $\mathbb{R}^n$ is contained in the set of roots of B_n, it is also effective. So it is a Coxeter group. The roots of B_n are obtained by applying the elements of B_n to the roots of a generating set. Therefore, as each element of B_n is either a permutation or some sign changes of the standard basis vectors, the root system is:

$$\Delta = \{\pm e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : i \neq j 1 \leq i, j \leq n\}.$$

Again, we want a $t = (a_1, \ldots, a_n)$ such that $r_1, \ldots, r_n$ are all positive with respect to t, i.e.,

$$\begin{aligned}
(t, r_1) &= (t, e_1) = a_1 > 0, \\
(t, r_2) &= (t, e_2 - e_1) = a_2 - a_1 > 0, \\
&\cdots \\
(t, r_n) &= (t, e_n - e_{n-1}) = a_n - a_{n-1} > 0.
\end{aligned}$$

Thus, we need

$$0 < a_1 < \cdots < a_n.$$

Such an element t certainly exists. We find that

$$\Delta^+ = \{e_1, \ldots, e_n\} \cup \{e_i - e_j : 1 \leq j < i \leq n\} \cup \{e_i + e_j : i \neq j, 1 \leq i, j \leq n\}.$$

And

$$\Delta^- = \{-e_1, \ldots, -e_n\} \cup \{-e_i + e_j : 1 \leq j < i \leq n\} \cup \{-e_i - e_j : i \neq j, 1 \leq i, j \leq n\}.$$

We have

$$\begin{aligned}
e_i &= r_i + \cdots + r_2 + r_1, \\
e_i - e_j &= r_i + \cdots + r_{j+1}, \\
e_i + e_j &= (r_i + \cdots + r_2 + r_1) + (r_j + \cdots + r_2 + r_1).
\end{aligned}$$

It follows that

$$\Pi = \{r_1, \ldots, r_n\}$$

is the base. Finally, we have

$$\frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = \begin{cases} -1/\sqrt{2} = -\cos(\pi/4) & \text{if} i = 1, j = 2, \\ -1/2 = -\cos(\pi/3) & \text{if } j = i+1,\ i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the Coxeter graph of B_n is:

Next, we consider D_n. Again, we will work inside $\mathrm{O}(\mathbb{R}^n)$, and the analysis will be similar to the case of B_n, except that we will change the notation somewhat. Let

$$L_n = \{f_J \in K_n : |J| \text{ is even}\}.$$

The set L_n is a subgroup of K_n because

$$|J \Delta L| = |J| + |L| - 2|J \cap L|.$$

The formula from above shows that S_n normalizes L_n, so we can consider the semi-direct product

$$D_n = L_n \rtimes S_n.$$

Let

$$r_1 = e_1 + e_2, r_2 = e_2 - e_1, \ldots, r_n = e_n - e_{n-1}.$$

We let $S_1, \ldots, S_n$ be the reflections along $r_1, \ldots, r_n$, respectively. Thus,

$$S_2 = (e_1 e_2), \ldots, S_n = (e_{n-1} e_n).$$

These elements, as before, generate S_n. What about the element S_1? We have

$$(e_1 + e_2)^{\perp} = \mathbb{R}(e_1 - e_2) + \mathbb{R}e_3 + \cdots + \mathbb{R}e_n.$$

Thus,

$$S_1(e_1 + e_2) \quad = \quad -(e_1 + e_2),$$

$$\begin{aligned} S_1(e_2 - e_1) &= e_2 - e_1, \\ S_1 e_3 &= e_3, \\ &\dots \\ S_1 e_n &= e_n. \end{aligned}$$

Thus, in fact,

$$\begin{aligned} S_1 e_1 &= -e_2, \\ S_1 e_2 &= -e_1. \end{aligned}$$

We see that

$$S_1 = f_{\{e_1,e_2\}} S_2.$$

Thus, S_1 is also in D_n. We assert that $S_1, \dots, S_n$ generate D_n. To prove this, it suffices to prove that the elements $f_{\{e_i,e_j\}}$ for $i \neq j$ are contained in D_n. There exists an element T in S_n such that $T\{e_1, e_2\} = \{e_i, e_j\}$. We have

$$T f_{\{e_1,e_2\}} T^{-1} = f_{\{e_i,e_j\}}.$$

This element is contained in the subgroup generated by $S_1, \dots, S_n$. We obtain that D_n is generated by the reflections $S_1, \dots, S_n$; it is also clear that D_n is effective, as $r_1, \dots, r_n$ is a basis. Hence, D_n is a Coxeter group. To find all the roots of D_n we apply the elements of D_n to the roots $r_1, \dots, r_n$ of our generating set $S_1, \dots, S_n$. This gives us:

$$\Delta = \{\pm e_i \pm e_j : 1 \leq j < i \leq n\}.$$

We want $r_1, \dots, r_n$ to be positive. Therefore, we need, letting $t = (a_1, \dots, a_n)$,

$$\begin{aligned} (r_1, t) &= a_1 + a_2 > 0, \\ (r_2, t) &= a_2 - a_1 > 0, \\ &\dots \\ (r_n, t) &= a_n - a_{n-1} > 0. \end{aligned}$$

That is, we need

$$a_n > \dots > a_2 > -a_1, a_1.$$

Such an element certainly exists. We can choose $a_1 > 0$, so that

$$a_n > \dots > a_2 > a_1 > 0.$$

The positive roots are then:

$$\Delta^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\}.$$

And the negative roots are:

$$\Delta^- = \{-e_i \pm e_j : 1 \le i < j \le n\}.$$

We note that

$$\begin{aligned} e_2 &= (1/2)r_1 + (1/2)r_2, \\ e_3 &= r_3 + (1/2)r_1 + (1/2)r_2, \\ &\cdots \\ e_n &= r_n + \cdots + r_3 + (1/2)r_1 + (1/2)r_2. \end{aligned}$$

This implies that for $1 \le j < i \le n$, $e_i \pm e_j$ is a nonnegative linear combination of elements of $\{r_1, \ldots, r_n\}$. Hence,

$$\Pi = \{r_1, \ldots, r_n\}$$

is the base. Finally, we have

$$\frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = \begin{cases} -1/2 = -\cos(\pi/3) & \text{if } j = i+1,\ i \ge 2, \\ -1/2 = -\cos(\pi/3) & \text{if } i = 1, j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the Coxeter graph of D_n is:

i) Find a set of mutually obtuse vectors $\{r_1, \ldots, r_n\}$ in $\mathbb{R}^n$ which form a basis such that these vectors have Coxeter graph of the desired type.

ii) Let S_i be the reflection determined by r_i; set $G = \langle S_1, \ldots, S_n \rangle$ in $\mathrm{O}(\mathbb{R}^n)$; the goal is to prove:

   a) G is finite;

   b) $\{r_1, \ldots, r_n\}$ is a base for G.

   This would prove that G is a Coxeter group with the desired graph. Note that G is already effective because $\{r_1, \ldots, r_n\}$ is a basis.

iii) The idea is to produce a *finite* set Γ^+ of vectors such that if we set $\Gamma = \Gamma^+ \cup -\Gamma^+$ then

a) $S_i\Gamma = \Gamma$ for $i = 1, \ldots, n$.

b) $\{r_1, \ldots, r_n\} \subset \Gamma^+$, and every vector in Γ is a root for G;

c) Every vector in Γ^+ is a nonnegative linear combination of $r_1, \ldots, r_n$.

From the first statement it follows that G is finite: G acts on the finite set Γ^* with no fixed points. Hence, G is a Coxeter group. From the second statement, Γ is in fact the root system of G: every element of Γ is contained in Δ, and as all the roots of G are obtained by applying the elements of G to $\{r_1, \ldots, r_n\}$, Δ is in Γ by the first statement. By the last statement, $\{r_1, \ldots, r_n\}$ is a base: we can always choose t such that $(t, r_i) > 0$ (proof: pick t such that t is perpendicular to $r_1 - r_2, r_1 - r_3, \ldots, r_1 - r_n$, so that $(t, r_1) = (t, r_2) = \cdots = (t, r_n)$; then multiply, if necessary, t by a scalar so that these numbers are positive).

iv) The problem is thus reduced to finding Γ^+. The algorithm for defining Γ^+ is as follows. Let $\Gamma_0 = \{r_1, \ldots, r_n\}$. For each $r \in \Gamma_0$ with $(r_1, r) < 0$ (note the strict inequality), obtain $S_1 r$. Note that this is a root for G. Do this for the other pairs r_2, S_2 through r_n, S_n. Let Γ_1 be the set of vectors obtained in this way, along with Γ_0. Now do the procedure again, applying it to the elements r of $\Gamma_1 - \Gamma_0$, and obtaining Γ_2. We get a chain of sets:

$$\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \ldots.$$

We note that every r in these sets is a root of G, and is a nonnegative linear combination of elements of $\{r_1, \ldots, r_n\}$. This is clear for the elements of Γ_0; assume it is true for the elements of Γ_j. Let $r \in \Gamma_{j+1}$. Then $r = S_i r'$ for some $r' \in \Gamma_j - \Gamma_{j-1}$ with $(r', r_i) < 0$. We have

$$r = S_i r' = r' + (-2\frac{(r_i, r')}{(r_i, r_i)})r_i.$$

Hence, r is a nonnegative linear combination of elements of $\{r_1, \ldots, r_n\}$. The *first key point* is to choose $r_1, \ldots, r_n$ so that *this algorithm terminates*, i.e., for some k we have $(r, r_i) \geq 0$ for all i and all $r \in \Gamma_k - \Gamma_{k-1}$. We then set

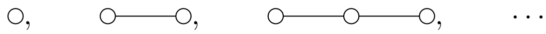
$$\Gamma^+ = \Gamma_k.$$

The *second key point* is to prove $S_i\Gamma = \Gamma$ for all i.

□

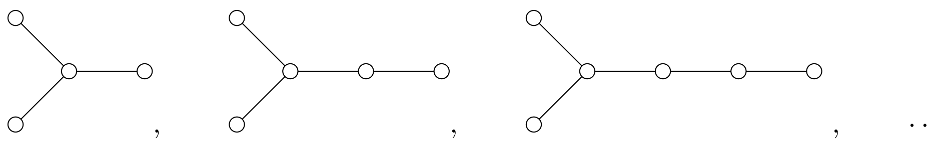
The Coxeter graphs of irreducible Coxeter groups:

A_n, $n \geq 1$:

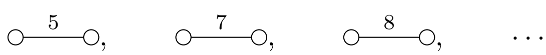


B_n, $n \geq 2$:

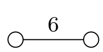
D_n, $n \geq 4$



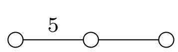
H_2^n, $n \geq 5$, $n \neq 6$:



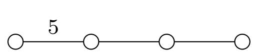
G_2:



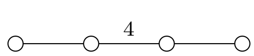
I_3:



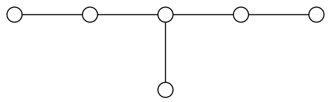
I_4:



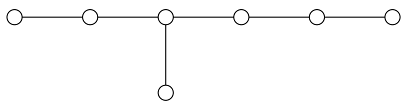
F_4:



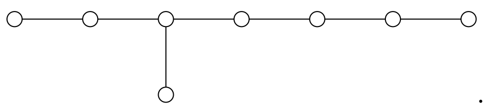
E_6:



E_7:



E_8:

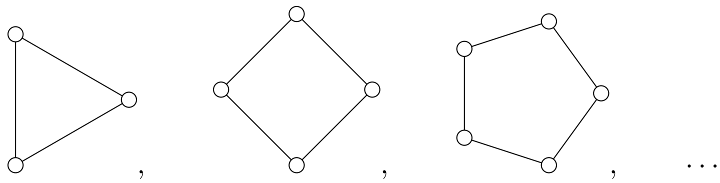


Graph	Base
A_n	$r_i = e_{i+1} - e_i$, $1 \leq i \leq n$ (V is the subspace of $\mathbb{R}^{n+1}$ spanned these n vectors.)
B_n	$r_1 = e_1$, $r_i = e_i - e_{i-1}$, $2 \leq i \leq n$
D_n	$r_1 = e_1 + e_2$, $r_i = e_i - e_{i-1}$, $2 \leq i \leq n$
H_2^n	$r_1 = (1, 0)$, $r_2 = (-\cos(\pi/n), \sin(\pi/n))$
G_2	$r_1 = e_2 - e_1$, $r_2 = e_1 - 2e_2 + e_3$
I_3	$r_1 = \beta(2\alpha + 1, 1, -2\alpha), r_2 = \beta(-2\alpha - 1, 1, 2\alpha), r_3 = \beta(2\alpha, -2\alpha - 1, 1)$
I_4	$r_1 = \beta(2\alpha + 1, 1, -2\alpha, 0), r_2 = \beta(-2\alpha - 1, 1, 2\alpha, 0)$ $r_3 = \beta(2\alpha, -2\alpha - 1, 1, 0), r_4 = \beta(-2\alpha, 0, -2\alpha - 1, 1)$ Here $\alpha = \cos(\pi/5) = \dfrac{1 + \sqrt{5}}{4}, \beta = \cos(2\pi/5) = \dfrac{-1 + \sqrt{5}}{4}$
F_4	$r_1 = -(1/2)(e_1 + e_2 + e_3 + e_4), r_2 = e_1, r_3 = e_2 - e_1, r_4 = e_3 - e_2$
E_6	$r_1 = (1/2)(\sum_{i=1}^{3} e_i - \sum_{i=4}^{8} e_i)$, $r_i = e_i - e_{i-1}$, $2 \leq i \leq 6$
E_7	$r_1 = (1/2)(\sum_{i=1}^{3} e_i - \sum_{i=4}^{8} e_i)$, $r_i = e_i - e_{i-1}$, $2 \leq i \leq 7$
E_8	$r_1 = (1/2)(\sum_{i=1}^{3} e_i - \sum_{i=4}^{8} e_i)$, $r_i = e_i - e_{i-1}$, $2 \leq i \leq 8$

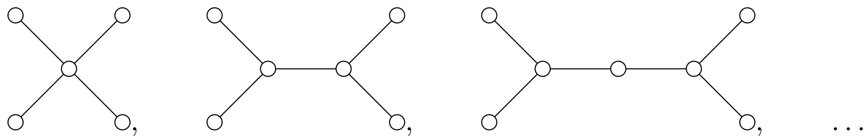
Graph	order of the group
A_n	$(n+1)!$
B_n	$2^n n!$
D_n	$2^{n-1} n!$
H_2^n	$2n$
G_2	12
F_4	$2^7 \cdot 3^2 = 1152$
I_3	$2^3 \cdot 3 \cdot 5 = 120$
I_4	$2^6 \cdot 3^2 \cdot 5^2 = 14400$
E_6	$2^7 \cdot 3^4 \cdot 5 = 51840$
E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2903040$
E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600$

Some marked graphs that are not positive definite:

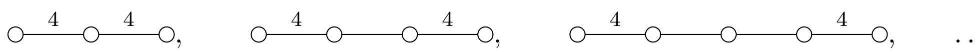
P_n, $n \geq 3$:



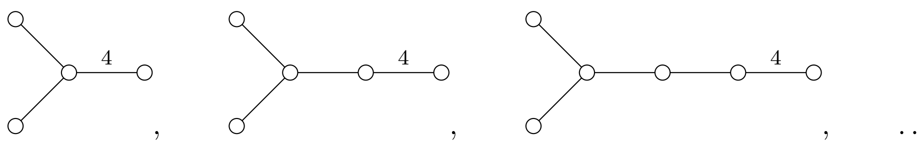
Q_n, $n \geq 5$:



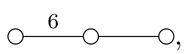
S_n, $n \geq 3$:



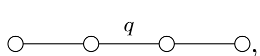
T_n, $n \geq 4$:



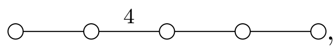
U_3:



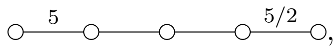
Z_4, $\cos(\frac{\pi}{q}) = \frac{3}{4}$, $4 < q < 5$:



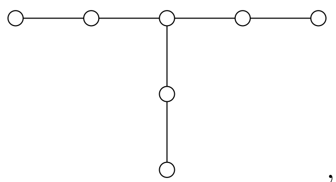
V_5:



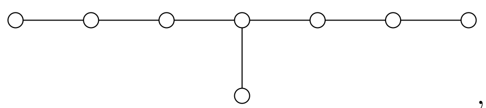
Y_5:



R_7:



R_8:



R_9: